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Macdonald operators
and
homological invariants of the colored Hopf link

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Abstract

Using a power sum (boson) realization for the Macdonald operators, we investigate the Gukov, Iqbal, Kozçaz and Vafa (GIKV) proposal for the homological invariants of the colored Hopf link, which include Khovanov-Rozansky homology as a special case. We prove the polynomiality of the invariants obtained by GIKV's proposal for arbitrary representations. We derive a closed formula of the invariants of the colored Hopf link for antisymmetric representations. We argue that a little amendment of GIKV's proposal is required to make all the coefficients of the polynomial non-negative integers.

1 Introduction and notation

In the setup of string theory, the invariants of the colored Hopf link are identified as topological open string amplitudes on the deformed conifold T^*S^3 as follows: the $U(N)$ Chern-Simons theory is realized by topological string on T^*S^3 with N topological D-branes wrapping on the base Lagrangian submanifold S^3. The Hopf link in S^3 consisting of two knots $\mathcal{K}_1$ and $\mathcal{K}_2$ can be introduced by a pair of new D-branes wrapping on Lagrangian three-cycles $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $S^3 \cap \mathcal{L}_i = \mathcal{K}_i$ [1]. The topological open string amplitude of this brane system is supposed to give the invariants of the Hopf link. The coloring or the representation attached to each knot $\mathcal{K}_i$ is related to the boundary states of the open string ending on $\mathcal{L}_i$ by the Frobenius relation. After geometric transition or by the large N duality [2, 3, 4], this brane configuration is mapped to the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$. The D-branes wrapping on S^3 disappear, but a pair of Lagrangian D-branes remains as a remnant of the Hopf link. We can describe the resulting D-brane system in terms of the toric diagram and compute the corresponding amplitude by the method of a topological vertex [5, 6].

The aim of this paper is to investigate the conjecture of Gukov, Iqbal, Kozçaz and Vafa (GIKV) [7] on the superpolynomial $\overline{\mathcal{P}}_{\lambda,\mu}(\mathbf{a}, \mathbf{q}, \mathbf{t})$ of the homological invariants of the Hopf link colored by two representations λ and μ. The superpolynomial of our interest is a polynomial in $(\mathbf{a}, \mathbf{q}, \mathbf{t}) \in \mathbb{C}^3$, such that a specialization $\mathbf{a} = \mathbf{q}^N$ leads the Poincaré polynomial of the $\mathfrak{sl}(N)$ link homology $\overline{\mathcal{P}}_{\mathfrak{sl}(N);\lambda,\mu}(\mathbf{q}, \mathbf{t})$, which is a two parameter $(\mathbf{q}, \mathbf{t})$ version of the $\mathfrak{sl}(N)$ link invariants. When the coloring is the N dimensional defining representation, it is called the Khovanov-Rozansky homology [8]. In [7], they argued the relation of the homological invariants with the refined topological vertex [9, 10] and the superpolynomial $\overline{\mathcal{P}}_{\lambda,\mu}(\mathbf{a}, \mathbf{q}, \mathbf{t})$ was expressed as a summation over all the partitions (see section 4). However, it is totally unclear whether the proposed expression actually gives a polynomial in $\mathbf{a}$ and a specialization $\mathbf{a} = \mathbf{q}^N$ leads the polynomial in $\mathbf{q}$ and $\mathbf{t}$. Moreover, the coefficients of the polynomial after the specialization are expected to be (non-negative) integers due to the following reason. In [11], it was argued that homological link invariants are related to a refinement of the BPS state counting in topological open string theory. The GIKV conjecture on homological link invariants of the Hopf link was based on this proposal. This means for the Hopf link L that there is a doubly graded homology theory $\mathcal{H}_{i,j}^{\mathfrak{sl}(N);\lambda,\mu}(L)$ whose graded Poincaré polynomial is

$$\overline{\mathcal{P}}_{\mathfrak{sl}(N);\lambda,\mu}(\mathbf{q}, \mathbf{t}) = \sum_{i,j \in \mathbb{Z}} \mathbf{q}^i \mathbf{t}^j \dim \mathcal{H}_{i,j}^{\mathfrak{sl}(N);\lambda,\mu}(L) \ . \tag{1.1}$$

The physical interpretation of $\mathcal{H}_{i,j}^{\mathfrak{sl}(N);\lambda,\mu}(L)$ as the Hilbert space of BPS states leads to a prediction on the dependence of the link homologies on the rank $N-1$. It has been conjectured [12, 13] that there exists a superpolynomial $\overline{\mathcal{P}}_{\lambda,\mu}(\mathbf{a},\mathbf{q},\mathbf{t})$ which is a rational function in three variables such that

$$\overline{\mathcal{P}}_{\mathfrak{sl}(N);\lambda,\mu}(\mathbf{q},\mathbf{t}) = \overline{\mathcal{P}}_{\lambda,\mu}(\mathbf{a} = \mathbf{q}^N, \mathbf{q}, \mathbf{t}) . \tag{1.2}$$

We see that (1.1) and (1.2) imply that the specialization $\mathbf{a} = \mathbf{q}^N$ leads to a polynomial with (non-negative) integer coefficients.

In this paper, we prove the following.

(i) The GIKV's proposal $\overline{\mathcal{P}}_{\lambda,\mu}^{\mathrm{GIKV}}(\mathbf{a},\mathbf{q},\mathbf{t})$ for the superpolynomial $\overline{\mathcal{P}}_{\lambda,\mu}(\mathbf{a},\mathbf{q},\mathbf{t})$ of the homological invariants of the colored Hopf link gives really a polynomial in $\mathbf{a}$ for arbitrary representations.

(ii) $\overline{\mathcal{P}}_{\lambda,1^s}^{\mathrm{GIKV}}(\mathbf{q}^N,\mathbf{q},\mathbf{t})$ vanishes for sufficiently small $N \in \mathbb{N}$, if one representation is anti-symmetric and the other is arbitrary.

(iii) $\overline{\mathcal{P}}_{1^r,1^s}^{\mathrm{GIKV}}(\mathbf{q}^N,\mathbf{q},\mathbf{t})$ with arbitrary $N \in \mathbb{N}$ is a polynomial in $\mathbf{q}$ and $\mathbf{t}$ with non-negative integer coefficients, if both representations are antisymmetric.

Furthermore, we perform the summation over the partitions and show a closed formula without assuming the condition $|t| < 1$ in [14] for the superpolynomial of the homological invariants for antisymmetric representations.

The key ingredients of our proof are the Macdonald operators and a non-standard scalar product for the Hall-Littlewood polynomials. The Macdonald operators are a set of difference operators which commute with each other on the space of symmetric functions. Their simultaneous eigenfunctions are the Macdonald polynomials which are two parameter (q,t) deformation of the Schur polynomials.[1] By using a power sum realization for the Macdonald operators [15, 16],[2] we investigate the polynomiality and integrality of the superpolynomial of the homological invariants obtained by GIKV's proposal.

On the other hand, the Hall-Littlewood polynomial is the symmetric polynomial obtained from the Macdonald polynomial by letting $q = 0$. The higher Macdonald operator is realized by using a (non-standard) scalar product for which the Hall-Littlewood polynomials are the basis dual to dominant monomials [17]. We show the pairwise orthogonality

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[1]We also use another set of parameters (Q,q,t) defined by $(\mathbf{a};\mathbf{q},\mathbf{t}) = (Q^{-\frac{1}{2}};t^{-\frac{1}{2}},-(t/q)^{\frac{1}{2}})$.

[2]We use a slightly different definition from that in [16].

of the Hall-Littlewood polynomials for this scalar product and use it for the calculation of the homological invariants.

For non-antisymmetric representations, the GIKV's proposal for the superpolynomial of the homological invariants calls for some improvement because it has negative integer coefficients in general. We find that this positivity problem may be overcome by replacing the Schur function by the Macdonald function with an appropriate specialization.

The paper is organized as follows. In section 2, we discuss two kinds of scalar product for the Hall-Littlewood polynomials, which are used in the following sections. In section 3, we give a power sum realization for the Macdonald operators which are defined in appendix C. Our main results are stated and proved in section 4. We review the Gukov *et al*'s conjectures on the homological invariants of the colored Hopf link and prove some of them. Section 5 and appendix F are devoted to the discussion on the positivity problem and its example, respectively. Appendix A contains a brief summary of the symmetric functions. Proofs of the key theorems are shown in appendices B and D. In appendix E, we comment on a relation between the torus knot and the Macdonald polynomial.

Notations. The following notations are used through this paper. Let λ be a Young diagram, i.e. a partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, which is a sequence of non-negative integers such that $\lambda_i \geq \lambda_{i+1}$ and $|\lambda| := \sum_i \lambda_i < \infty$. $\lambda^\vee$ is its conjugate (dual) diagram. $\ell(\lambda) := \lambda_1^\vee$ is the length. Let $p = (p_1, p_2, \cdots)$ be the power sum symmetric functions in $x = (x_1, x_2, \cdots)$ defined as $p_n(x) := \sum_{i \geq 1} x_i^n$. We treat any symmetric function $f(x)$ in x as a function in p unless otherwise stated, and sometimes denote it as $f(x(p))$. $P_\lambda(x; q, t)$, $P_\lambda(x; t)$, $s_\lambda(x)$ and $e_\lambda(x)$ are the Macdonald, the Hall-Littlewood, the Schur and the elementary symmetric function in x, respectively. We define the following specialization:

$$p_n(q^\lambda t^\rho) := \sum_{i=1}^{\ell(\lambda)} (q^{n\lambda_i} - 1) t^{n(\frac{1}{2} - i)} + \frac{1}{t^{\frac{n}{2}} - t^{-\frac{n}{2}}} = \sum_{i=1}^N q^{n\lambda_i} t^{n(\frac{1}{2} - i)} + \frac{t^{-nN}}{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}, \qquad (1.3)$$

which is independent of N for any $N \geq \ell(\lambda)$. We do not have to assume that $|t| > 1$. Let $p_n(x, y) := p_n(x) + p_n(y)$; then,

$$p_n(cq^\lambda t^\rho, Lt^{-\rho}) = c^n \sum_{i=1}^{\ell(\lambda)} (q^{n\lambda_i} - 1) t^{n(\frac{1}{2} - i)} + \frac{c^n - L^n}{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}, \qquad c, L \in \mathbb{C}. \qquad (1.4)$$

For $N \in \mathbb{Z}$ and $N \geq \ell(\lambda)$, $p_n(q^\lambda t^{N+\rho}, t^{-\rho}) = \sum_{i=1}^N q^{n\lambda_i} t^{n(N+\frac{1}{2} - i)}$ is the power sum in N variables $\{q^{\lambda_i} t^{N+\frac{1}{2} - i}\}_{1 \leq i \leq N}$. We do not have to use any analytic continuation or approximation because we treat any functions as formal power series. For example,

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$\Pi(x, y; q, t) := \exp\left\{\sum_{n>0} \frac{1}{n} \frac{1-t^n}{1-q^n} p_n(x) p_n(y)\right\}$ and $\Delta(x; t) := \exp\left\{-\sum_{n>0} \frac{1-t^n}{n} \sum_{i<j} \frac{x_i^n}{x_j^n}\right\}$ are power series in formal variables $p_n(x)$, $p_n(y)$ and x_i respectively. We also let $g_\lambda := \prod_{(i,j)\in\lambda} (-1) q^{\lambda_i - j} t^{-\lambda_j^\vee + i}$ and $v := (q/t)^{\frac{1}{2}}$. The q-integer $[\, N \,]_t := \frac{1-t^N}{1-t}$ and the q-binomial coefficient $\begin{bmatrix} N \\ r \end{bmatrix}_t := \prod_{i=1}^r \frac{1-t^{N-r+i}}{1-t^i}$ are polynomials in t with non-negative integer coefficients for $N, r \in \mathbb{N}$.

2 Scalar product for the Hall-Littlewood polynomial

First, we discuss two kinds of scalar product for symmetric polynomials, which we will use later. In this section, we treat symmetric functions as symmetric polynomials in finite number of variables $x = (x_1, \cdots, x_N)$ by setting $x_i = 0$ for $i \geq N + 1$. For a partition $\lambda = (\lambda_1, \cdots, \lambda_N)$, we denote $x^\lambda := x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_N^{\lambda_N}$ and $m_j := \#\{\, \lambda_i \mid \lambda_i = j \,\}$, i.e. $\lambda = (0^{m_0} 1^{m_1} 2^{m_2} \cdots)$ with $\sum_{i\geq 0} m_i = N$. The Hall-Littlewood polynomials $P_\lambda(x; t)$ are obtained from the Macdonald polynomials $P_\lambda(x; q, t)$ defined in appendix A by letting $q = 0$:

$$\begin{aligned}
P_\lambda(x; t) &:= P_\lambda(x; 0, t) \\
&= v_\lambda^{-1}(t) \sum_{\sigma \in \mathcal{S}_N} \sigma\left(x^\lambda \Delta(\bar{x}; t)^{-1}\right), \qquad v_\lambda(t) := \prod_{j\geq 0} [\, m_j \,]!_t \\
\Delta(x; t) &:= \exp\left\{-\sum_{n>0} \frac{1-t^n}{n} \sum_{i<j} \frac{x_i^n}{x_j^n}\right\} = \prod_{i<j}\left(1 - \frac{x_i}{x_j}\right) \sum_{n\geq 0}\left(\frac{tx_i}{x_j}\right)^n
\end{aligned} \tag{2.1}$$

with the symmetric group $\mathcal{S}_N$ on N variables.[3] Here $\bar{x} := (x_N, x_{N-1}, \cdots, x_1)$, $[\, N \,]!_t := [\, 1 \,]_t [\, 2 \,]_t \cdots [\, N \,]_t$ and $[\, N \,]_t := (1 - t^n)/(1 - t)$. The canonical scalar product $\langle *, * \rangle_t$ is defined from $\langle *, * \rangle_{q,t}$ in appendix A by

$$\langle P_\lambda(x; t), P_\mu(x; t) \rangle_t := \langle P_\lambda(x; 0, t), P_\mu(x; 0, t) \rangle_{0,t} = \delta_{\lambda,\mu} \prod_{j\geq 1} \prod_{i=1}^{m_j} \frac{1}{1 - t^i}. \tag{2.2}$$

We abbreviate it to $\langle P_\lambda, P_\mu \rangle_t$. The Cauchy formula (A.5) is now

$$\sum_\lambda \frac{1}{\langle P_\lambda, P_\lambda \rangle_t} P_\lambda(x; t) P_\lambda(y; t) = \exp\left\{\sum_{n>0} \frac{1-t^n}{n} p_n(x) p_n(y)\right\} \tag{2.3}$$

which is summed over all partitions λ.

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[3] Note that $\Delta(x; t) = \prod_{i<j} \frac{1 - x_i/x_j}{1 - t x_i/x_j}$ by the analytic continuation.

For functions f and g in x, let us define a second scalar product as [17]

$$\langle f(x), g(x) \rangle''_{N;t} := \oint \prod_{j=1}^{N} \frac{dx_j}{2\pi i x_j} f(\bar{x}^{-1}) \Delta(x;t) g(x). \tag{2.4}$$

Here x_j's are formal parameters, and $\exp f$ is defined as the Taylor expansion in f. For a Laurent series $f(x)$ in x, $\oint \frac{dx}{2\pi i x} f(x)$ denotes the constant term in f, i.e. $\oint \frac{dx}{2\pi i x} \sum_{n \in \mathbb{Z}} f_n x^n = f_0$. Note that the kernel function $\Delta(x;t)$ is not symmetric in x_i. Then, $\langle f(x), g(x) \rangle''_{N;t} = \langle g(\bar{x}), f(\bar{x}) \rangle''_{N;t}$. Hence, the second scalar product is symmetric only for the symmetric functions. The Hall-Littlewood polynomials $P_\lambda(x;t)$ with $\ell(\lambda) \leq N$ are pairwise orthogonal for the second scalar product and we have [4]

Theorem 2.1.

$$\langle P_\lambda(x;t), P_\mu(x;t) \rangle''_{N;t} = \langle P_\mu(x;t), P_\lambda(x;t) \rangle''_{N;t} = \delta_{\lambda,\mu} v_\lambda(t)^{-1} [N]!_t. \tag{2.5}$$

A proof is given in appendix B. Since $P_{1^r}(x;t)$ coincides with the elementary symmetric polynomial $e_r(x)$, we obtain $\langle P_\lambda(x;t), e_r(x) \rangle''_{N;t} = \delta_{\lambda,1^r} \begin{bmatrix} N \\ r \end{bmatrix}_t$, with $\begin{bmatrix} N \\ r \end{bmatrix}_t := \prod_{i=1}^{r} \frac{1-t^{N-r+i}}{1-t^i}$. Note that from (A.12) it follows that

$$\frac{\langle e_r, e_r \rangle''_{N;t}}{\langle e_r, e_r \rangle_t} = (-1)^r t^{\frac{r}{2}} \frac{e_{N-r}(t^\rho)}{e_N(t^\rho)}, \qquad e_r(t^\rho) = (-1)^r t^{\frac{r}{2}} \langle e_r, e_r \rangle_t, \tag{2.6}$$

which we will use later.

3   Power sum realization for Macdonald operators

In this section, we give a power sum (boson) realization for the Macdonald operators which are defined in appendix C. Let $p = (p_1, p_2, \cdots)$ be the power sum symmetric functions in infinite number of variables $x = (x_1, x_2, \cdots)$ defined as $p_n(x) := \sum_{i \geq 1} x_i^n$. We treat any symmetric function $f(x)$ in x as a function in p unless otherwise stated.

3.1   Macdonald operators by the power sum

First, we define a commutative family of the difference operators in power sums, whose eigenfunctions are the Macdonald functions. Let $z = (z_1, \cdots, z_r)$. For $r = 0, 1, 2, \cdots$, let H and H^r be $H := \sum_{r \geq 0} w^r H^r$, $H^0 := 1$ and

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[4]It was shown in [17] that the Hall-Littlewood polynomials are the basis dual to dominant monomials.

$$H^r := e_r(t^\rho) \oint \prod_{\alpha=1}^r \frac{dz_\alpha}{2\pi i z_\alpha} \Delta(z; t^{-1}) \varphi^r(z), \qquad \varphi^r(z) := \varphi_+^r(z) \varphi_-^r(z),$$

$$\varphi_+^r(z) := \exp\left\{ \sum_{n>0} \frac{1-t^{-n}}{n} \sum_{\alpha=1}^r z_\alpha^n p_n \right\},$$

$$\varphi_-^r(z) := \exp\left\{ \sum_{n>0} \frac{1-t^{-n}}{n} t^n \sum_{\alpha=1}^r z_\alpha^{-n} p_n^* \right\} = \exp\left\{ \sum_{n>0} (q^n-1) \sum_{\alpha=1}^r z_\alpha^{-n} \frac{\partial}{\partial p_n} \right\}, \quad (3.1)$$

with $e_r(t^\rho) = \prod_{i=1}^r \frac{t^{\frac{1}{2}}}{t^i - 1}$ from (A.12) and $p_n^* := \frac{1-q^n}{1-t^n} n \frac{\partial}{\partial p_n}$. Here z_α and p_n are formal variables and $\oint \frac{dz}{2\pi i z} f(z)$ denotes the constant term in f. Note that H^r is written by the second scalar product (2.4) as $H^r = e_r(t^\rho) \langle \varphi_+^r(z^{-1}), \varphi_-^r(z) \rangle''_{r;t^{-1}}$. We also denote them as $H(x)$, $H^r(x)$ and $\varphi(z; x)$ if they act on the power sums $p_n(x)$ in x, but they are independent of the number of variables x_i.

The Cauchy formula (2.3) for the Hall-Littlewood function leads to

$$\varphi^r(z) = \sum_{\substack{\lambda \\ \ell(\lambda) \le r}} \frac{P_\lambda(x(p); t^{-1}) P_\lambda(z; t^{-1})}{\langle P_\lambda, P_\lambda \rangle_{t^{-1}}} \sum_{\substack{\mu \\ \ell(\mu) \le r}} t^{|\mu|} \frac{P_\mu(z^{-1}; t^{-1}) P_\mu(x(p^*); t^{-1})}{\langle P_\mu, P_\mu \rangle_{t^{-1}}}. \qquad (3.2)$$

Here $P_\lambda(x(p); t)$ is the Hall-Littlewood function in terms of the power sums p and $\ell(\lambda) := \lambda_1^\vee$ is the length of λ. Thus, H^r is also realized by the Hall-Littlewood function as

$$H^r = e_r(t^\rho) \sum_{\substack{\lambda \\ \ell(\lambda) \le r}} P_\lambda(x(p); t^{-1}) \frac{t^{|\lambda|} \langle P_\lambda, P_\lambda \rangle''_{r;t^{-1}}}{\langle P_\lambda, P_\lambda \rangle_{t^{-1}}^2} P_\lambda(x(p^*); t^{-1}). \qquad (3.3)$$

Then, we have [16]([15] for $r=1$) [5]

Theorem 3.1. *The Macdonald function $P_\lambda(x; q, t)$ is an eigenfunction for H^r:*

$$H P_\lambda(x; q, t) = P_\lambda(x; q, t) E_\lambda,$$
$$H^r P_\lambda(x; q, t) = P_\lambda(x; q, t) e_r(q^\lambda t^\rho),$$
$$E_\lambda := \exp\left\{ -\sum_{n>0} \frac{(-w)^n}{n} p_n(q^\lambda t^\rho) \right\} = \sum_{r \ge 0} w^r e_r(q^\lambda t^\rho), \qquad (3.4)$$

and therefore, H^r commute with each other $[H^r, H^s] = 0$ on the space of symmetric functions.

The proof is given by comparing H^r with the Macdonald operators.

3.2 Action on the Cauchy kernel

Second, we discuss the properties of H^r, which we will use in subsection 4.3. For the Cauchy kernel $\Pi(x, y; q, t)$ in (A.5), we obtain

$$\frac{\varphi^r(z; x)\Pi(x, y; q, t)}{\Pi(x, y; q, t)} = \exp\left\{\sum_{n>0}\frac{1 - t^{-n}}{n}\left(p_n(x)p_n(z) + p_n(ty)p_{-n}(z)\right)\right\}, \tag{3.5}$$

with $p_n(z) = \sum_{\alpha=1}^r z_\alpha^n$, $(n \in \mathbb{Z})$. Since

$$\Delta(z; t^{-1})\varphi^r(z; x)\Pi(x, y; q, t) = \Delta(t\bar{z}^{-1}; t^{-1})\varphi^r(t\bar{z}^{-1}; y)\Pi(x, y; q, t), \tag{3.6}$$

with $\bar{z} := (z_r, z_{r-1}, \cdots, z_1)$, the following important duality holds for any variables $x = (x_1, x_2, \cdots)$ and $y = (y_1, y_2, \cdots)$, even though their numbers of components are different:

$$H^r(x)\Pi(x, y; q, t) = H^r(y)\Pi(x, y; q, t). \tag{3.7}$$

In a way similar to (3.2) and (3.3), we obtain

$$\frac{\varphi^r(z; x)\Pi(x, y; q, t)}{\Pi(x, y; q, t)} = \sum_{\substack{\lambda \\ \ell(\lambda)\leq r}}\frac{P_\lambda(x(p); t^{-1})P_\lambda(z; t^{-1})}{\langle P_\lambda, P_\lambda\rangle_{t^{-1}}}\sum_{\substack{\mu \\ \ell(\mu)\leq r}}\frac{P_\mu(z^{-1}; t^{-1})P_\mu(ty(p); t^{-1})}{\langle P_\mu, P_\mu\rangle_{t^{-1}}}, \tag{3.8}$$

$$\frac{H^r(x)\Pi(x, y; q, t)}{\Pi(x, y; q, t)} = e_r(t^\rho)\sum_{\substack{\lambda \\ \ell(\lambda)\leq r}}P_\lambda(x(p); t^{-1})\frac{\langle P_\lambda, P_\lambda\rangle''_{r;t^{-1}}}{\langle P_\lambda, P_\lambda\rangle^2_{t^{-1}}}P_\lambda(ty(p); t^{-1}). \tag{3.9}$$

When $x = t^{-\rho}$, we have[6]

Proposition 3.2.

$$\left.\frac{H^r(x)\Pi(x, y; q, t)}{\Pi(x, y; q, t)}\right|_{x=t^{-\rho}} = g_{1^r}e_r(y, t^{-\rho}), \tag{3.10}$$

with $\sum_{r\geq0}(-w)^r e_r(x, y) := \exp\left\{-\sum_{n>0}p_n(x, y)w^n/n\right\}$ and $g_{1^r} := (-1)^r t^{-\frac{r(r-1)}{2}}$.

Proof. From (3.9) and $P_\lambda(t^\rho; t) = \delta_{\lambda, 1^r}e_r(t^\rho)$, it follows that

$$\text{lhs} = e_r(t^\rho)\sum_{s=0}^r P_{1^s}(t^{-\rho}; t^{-1})\frac{\langle P_{1^s}, P_{1^s}\rangle''_{r;t^{-1}}}{\langle P_{1^s}, P_{1^s}\rangle^2_{t^{-1}}}P_{1^s}(ty(p); t^{-1}). \tag{3.11}$$

But from (2.6) and (A.12), we obtain

$$\text{lhs} = (-1)^r\prod_{i=1}^r t^{1-i}\sum_{s=0}^r e_{r-s}(t^{-\rho})e_s(y). \tag{3.12}$$

Then, $\sum_{s=0}^r e_{r-s}(x)e_s(y) = e_r(x, y)$ proves the proposition. $\qquad\square$

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[6] This proposition was proved in [14] by assuming $|t| < 1$. However, here we do not have to assume it.

3.3 Product of Macdonald operators

Next we consider the product of the H^r's which we will use in subsection 4.4. For a partition λ, let $\ell := \ell(\lambda)$, $H^\lambda := H^{\lambda_1} H^{\lambda_2} \cdots H^{\lambda_\ell}$, $\boldsymbol{z}^i = (\boldsymbol{z}_1^i, \boldsymbol{z}_2^i, \cdots, \boldsymbol{z}_{\lambda_i}^i)$ and $z := (z_1, \cdots, z_{|\lambda|}) := (\boldsymbol{z}_1^1, \cdots, \boldsymbol{z}_{\lambda_1}^1, \boldsymbol{z}_1^2, \cdots, \boldsymbol{z}_{\lambda_2}^2, \cdots, \boldsymbol{z}_1^\ell, \cdots, \boldsymbol{z}_{\lambda_\ell}^\ell)$; then,

$$\Delta(z;t) = \exp\left\{-\sum_{n>0}\frac{1-t^n}{n}\sum_{i<j}p_n(\boldsymbol{z}^i)p_{-n}(\boldsymbol{z}^j)\right\}\prod_{i=1}^\ell \Delta(\boldsymbol{z}^i;t). \tag{3.13}$$

By the OPE relations (as difference operators)

$$\varphi^s(w)\varphi^r(z) = \exp\left\{\sum_{n>0}\frac{(1-t^{-n})(q^n-1)}{n}p_{-n}(w)p_n(z)\right\}\varphi_+^s(w)\varphi_+^r(z)\varphi_-^s(w)\varphi_-^r(z) \tag{3.14}$$

with $z = (z_1, \cdots, z_r)$ and $w = (w_1, \cdots, w_s)$, we obtain

$$\frac{\varphi^{\lambda_\ell}(\boldsymbol{z}^\ell;x)\cdots\varphi^{\lambda_1}(\boldsymbol{z}^1;x)\Pi(x,y;q,t)}{\Pi(x,y;q,t)}$$

$$= \exp\left\{\sum_{n>0}\frac{1-t^{-n}}{n}\left\{p_n(x)p_n(z) + p_n(ty)p_{-n}(z) + (q^n-1)\sum_{i<j}p_n(\boldsymbol{z}^i)p_{-n}(\boldsymbol{z}^j)\right\}\right\}$$

$$= \exp\left\{\sum_{n>0}\frac{1-t^{-n}}{n}\left\{p_n(x)p_n(z) + p_n(ty)p_{-n}(z) + \sum_{i<j}p_n(q\boldsymbol{z}^i)p_{-n}(\boldsymbol{z}^j)\right\}\right\}$$

$$\times\frac{\Delta(z;t^{-1})}{\prod_{i=1}^\ell \Delta(\boldsymbol{z}^i;t^{-1})}. \tag{3.15}$$

When $(x,y) = (t^\rho, ct^{-\rho})$ with $c \in \mathbb{C}$, since $p_n(t^\rho) = (t^{\frac{n}{2}} - t^{-\frac{n}{2}})^{-1}$ it follows that

$$\prod_{i=1}^\ell \Delta(\boldsymbol{z}^i;t^{-1}) \times \left.\frac{\varphi^{\lambda_\ell}(\boldsymbol{z}^\ell;x)\cdots\varphi^{\lambda_1}(\boldsymbol{z}^1;x)\Pi(x,ct^{-\rho};q,t)}{\Pi(x,ct^{-\rho};q,t)}\right|_{x=t^\rho}$$

$$= \Delta(z;t^{-1})\exp\left\{\sum_{n>0}\frac{1-t^{-n}}{n}\sum_{i<j}p_n(q\boldsymbol{z}^i)p_{-n}(\boldsymbol{z}^j)\right\}\prod_{\alpha=1}^{|\lambda|}\frac{1-t^{\frac{1}{2}}c/z_\alpha}{1-t^{-\frac{1}{2}}z_\alpha}, \tag{3.16}$$

which is a polynomial of degree $|\lambda|$ in c. For abbreviation, we write $1/(1-z)$ instead of $\sum_{n\geq 0}z^n$. Therefore, by taking the constant term in z, we have

Proposition 3.3.

$$\left.\frac{H^\lambda(x)\Pi(x,ct^{-\rho};q,t)}{e_\lambda(t^\rho)\Pi(x,ct^{-\rho};q,t)}\right|_{x=t^\rho}$$

$$= \oint\prod_{\alpha=1}^{|\lambda|}\frac{dz_\alpha}{2\pi iz_\alpha}\frac{1-c/z_\alpha}{1-z_\alpha}\prod_{\alpha<\beta}^{|\lambda|}\frac{1-z_\alpha/z_\beta}{1-z_\alpha/tz_\beta}\prod_{i<j}^\ell\prod_{\alpha=1}^{\lambda_i}\prod_{\beta=1}^{\lambda_j}\frac{1-q\boldsymbol{z}_\alpha^i/\boldsymbol{z}_\beta^j}{1-q\boldsymbol{z}_\alpha^i/t\boldsymbol{z}_\beta^j} \tag{3.17}$$

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is a polynomial of degree $|\lambda|$ in c and a polynomial in q and $1/t$ with integer coefficients and vanishes when $c = 1$.

Proof. The rhs of (3.17) reduces to

$$\oint \prod_{\alpha=1}^{|\lambda|} \frac{dz_\alpha}{2\pi i z_\alpha} \left(1 - \frac{c}{z_\alpha}\right) \sum_{n=0}^{\alpha} (z_\alpha)^n \times \prod_{\alpha<\beta}^{|\lambda|} \left(1 - \frac{z_\alpha}{z_\beta}\right) \sum_{n=0}^{\alpha} \left(\frac{z_\alpha}{t z_\beta}\right)^n$$
$$\times \prod_{i<j}^{\ell} \prod_{\alpha=1}^{\lambda_i} \prod_{\beta=1}^{\lambda_j} \left(1 - \frac{q\boldsymbol{z}_\alpha^i}{\boldsymbol{z}_\beta^j}\right) \sum_{n=0}^{|\boldsymbol{z}_\alpha^i|} \left(\frac{q\boldsymbol{z}_\alpha^i}{t\boldsymbol{z}_\beta^j}\right)^n, \quad (3.18)$$

with $|\boldsymbol{z}_\alpha^i| := \alpha + \sum_{k=1}^{i-1} \lambda_k$, so this is a polynomial in q and $1/t$ with integer coefficients. When $c = 1$, since $(1 - 1/z_\alpha) \sum_{n\geq0} (z_\alpha)^n = 1/z_\alpha$, the integrand in (3.17) has no constant term in the last variable $z_{|\lambda|}$; thus, (3.17) vanishes. $\qquad \square$

Let

$$\widetilde{Z}^\lambda \prod_{j=1}^{\lambda_1} (1 - c/t^{j-1}) := \frac{H^\lambda(x)\Pi(x, ct^{-\rho}; q, t)}{e_\lambda(t^\rho)\Pi(x, ct^{-\rho}; q, t)}\Bigg|_{x=t^\rho}. \quad (3.19)$$

Then, for example,

$$\widetilde{Z}^{(3)} = 1,$$
$$\widetilde{Z}^{(2,1)} = (1 - cq) - c(1 - q)/t^2, \quad (3.20)$$
$$\widetilde{Z}^{(1,1,1)} = (1 - cq)\{(1 - cq^2) - c(1 - q)(1 + 2q)/t - c(1 - q)^2/t^2\} + c^2(1 - q)^2/t^3.$$

Direct calculation of the functions $\widetilde{Z}^\lambda$ for $|\lambda| \leq 7$ suggests the following conjecture.

Conjecture 3.4. *If $c = t^N$, $N \in \mathbb{Z}$ with $0 \leq N < \lambda_1$, then (3.17) vanishes.*

4 Homological link invariants from GIKV conjecture

In this section, we introduce the conjecture on homological link invariants of the colored Hopf link by Gukov, Iqbal, Kozçaz and Vafa [7] and prove some of them.[7]

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[7] Our $(Q, Q^{\mathrm{GIKV}}; q, t; \mathbf{q}, \mathbf{t})$ equals $(Q\sqrt{q_2/q_1}, Q; q_1^{-1}, q_2^{-1}; q, t)$ of [7].
In [7], $\bar{\mathcal{P}}_{\lambda,\mu}(\mathbf{a}; \mathbf{q}, \mathbf{t}) = (-\mathbf{a})^{|\lambda|+|\mu|}\mathbf{t}^{2|\lambda||\mu|}G_{\lambda,\mu}(-Q^{\mathrm{GIKV}}; \mathbf{q}, \mathbf{t})$ and $G_{\lambda,\mu}(-Q^{\mathrm{GIKV}}; \mathbf{q}, \mathbf{t}) = Z_{\lambda,\mu}^{\mathrm{inst}}(Q; q, t)$, where $(\mathbf{a}; \mathbf{q}, \mathbf{t}) := (Q^{-\frac{1}{2}}; t^{-\frac{1}{2}}, -(t/q)^{\frac{1}{2}})$. Note that $\mathbf{a} = \mathbf{q}^N$ and $\mathbf{t} = -1$ are equivalent to $Q = t^N$ and $q = t$, respectively.

4.1 GIKV conjecture

First, we recall the GIKV conjecture and present our main theorem. Following [7] let us consider

$$Z_{\lambda,\mu}(Q;q,t) := \sum_\eta s_\lambda(q^\eta t^\rho) s_\mu(q^\eta t^\rho) \prod_{(i,j)\in\eta} \frac{Q}{(1-q^{\eta_i-j+1}t^{\eta_j^\vee-i})(1-q^{-\eta_i+j}t^{-\eta_j^\vee+i-1})}. \quad (4.1)$$

Here $s_\lambda(q^\eta t^\rho)$ is the Schur function in the power sum $p_n(q^\eta t^\rho)$. From (A.10) and (A.8), it is written by the Macdonald functions $P_\lambda(x;q,t)$ defined in appendix A as follows:

$$\begin{aligned}
Z_{\lambda,\mu}(Q;q,t) &= \sum_\eta (-v^{-1}Q)^{|\eta|} P_{\eta^\vee}(q^\rho;t,q) P_\eta(t^\rho;q,t) s_\lambda(q^\eta t^\rho) s_\mu(q^\eta t^\rho) \\
&= \sum_\eta \frac{Q^{|\eta|}}{\langle P_\eta, P_\eta\rangle_{q,t}} P_\eta(t^{-\rho};q,t) P_\eta(t^\rho;q,t) s_\lambda(q^\eta t^\rho) s_\mu(q^\eta t^\rho). \quad (4.2)
\end{aligned}$$

Note that $Z_{\bullet,\bullet}(Q;q,t) = \Pi(Qt^\rho, t^{-\rho};q,t)$. Then, $Z_{\lambda,\mu}^{\text{inst}}(Q;q,t) := Z_{\lambda,\mu}(Q;q,t)/Z_{\bullet,\bullet}(Q;q,t)$ is a power series in Q and a meromorphic function in q and t. But $Z_{\lambda,\mu}^{\text{inst}}(Q;q,t)$ is expected to give the superpolynomial of the homological invariants and Gukov, Iqbal, Kozçaz and Vafa claimed the following results.

Conjecture 4.1[7].
(1) $Z_{\lambda,\mu}^{\text{inst}}(Q;q,t)$ *is a finite polynomial in* Q.
(2) $Z_{\lambda,\mu}^{\text{inst}}(t^N;q,t)$ *vanishes for sufficiently small* $N \in \mathbb{Z}_{\geq 0}$.
(3) $Z_{\lambda,\mu}^{\text{inst}}(t^N;q,t)(-1)^{|\lambda|+|\mu|}$ $(N \in \mathbb{Z}_{\geq 0})$ *is a finite polynomial in* q *and* t *with integer coefficients.*
(4) $Z_{\lambda,\mu}^{\text{inst}}(t^N;q,t)(-1)^{|\lambda|+|\mu|}$ *for sufficiently large* N *coincides with the* $sl(N)$ *homological invariants of the Hopf link colored by the representations* λ *and* μ *up to an overall factor.*

Note that if the conjecture 4.1.(1) is established, one can easily calculate $Z_{\lambda,\mu}^{\text{inst}}(t^N;q,t)$ for any given $(\lambda,\mu;N)$ and check (2) and (3). In the following subsections, we will show part of the GIKV conjecture by proving a series of propositions.[8]

Theorem 4.2.
(*i*) *Conjecture* (1) *holds for arbitrary* (λ,μ) (*proposition* 4.10).
(*ii*) *Conjecture* (2) *holds for arbitrary* (λ,μ) *with* $|\lambda|+|\mu| \leq 7$ *or* $\mu = 1^s$ (*propositions* 4.10 *and* 4.8).
(*iii*) *Conjecture* (3) *holds for* $(\lambda,\mu) = (\lambda,1^s)$ *with* $|\lambda| \leq 7$ *or* $\lambda = 1^r$ (*propositions* 4.8 *and* 4.6).
(*iv*) *Conjecture* (4) *holds for* $(\lambda,\mu) = (1^r,1)$ (*proposition* 4.6).

---

[8]In [14], we have shown (ii)–(iv) by assuming the condition $|t| < 1$.

4.2 $q = t$ case

When $q = t$, we have

Proposition 4.3.

$$\frac{Z_{\lambda,\mu}(Q;q,q)}{Z_{\bullet,\bullet}(Q;q,q)} = f_\lambda f_\mu s_\lambda(Qq^\rho, q^{-\rho}) s_\mu(Qq^\lambda q^\rho, q^{-\rho}), \tag{4.3}$$

which is a polynomial of degree $|\lambda| + |\mu|$ *in* Q*. Here* $f_\lambda := \prod_{(i,j)\in\lambda}(-1)q^{\lambda_i - \lambda_j^\vee + i - j}$.

Proof.   First, we have the following nontrivial identity:

$$s_\lambda(q^\rho) s_\mu(q^{\lambda+\rho}) = f_\lambda f_\mu \sum_\nu s_{\lambda/\nu}(q^{-\rho}) s_{\mu/\nu}(q^{-\rho}) = s_\mu(q^\rho) s_\lambda(q^{\mu+\rho}), \tag{4.4}$$

which is proved by $s_{\lambda/\mu}(q^\rho) = s_{\lambda^\vee/\mu^\vee}(-q^{-\rho})$ and the cyclic symmetry of the topological vertex [6]. Since the above identity and $s_\lambda(q^\rho) = f_\lambda s_{\lambda^\vee}(-q^\rho)$, it follows that

$$Z_{\lambda,\mu}(Q;q,q) = f_\mu \sum_{\eta,\nu} Q^{|\eta|} s_{\mu/\nu}(q^{-\rho}) s_{\eta/\nu}(q^{-\rho}) s_\lambda(q^\rho) s_\eta(q^{\lambda+\rho}). \tag{4.5}$$

The Cauchy formula (A.5) and the adding formula (A.4) yield

$$Z_{\lambda,\mu}(Q;q,q) = f_\mu s_\mu(Qq^{\lambda+\rho}, q^{-\rho}) \Pi(Qq^{\lambda+\rho}, q^{-\rho}; q, q) s_\lambda(q^\rho). \tag{4.6}$$

But ((2.12) and (5.20) of [21])

$$\frac{\Pi(Qq^{\lambda+\rho}, q^{-\rho}; q, q)}{\Pi(Qq^\rho, q^{-\rho}; q, q)} = f_\lambda \frac{s_\lambda(Qq^\rho, q^{-\rho})}{s_\lambda(q^\rho)}, \tag{4.7}$$

which completes the proof.   □

Note that for $N \in \mathbb{Z}$ and $N \geq \ell(\lambda)$, $p_n(q^{\lambda+N+\rho}, q^{-\rho}) = \sum_{i=1}^N q^{n(\lambda_i + i - \frac{1}{2})}$. Thus, $s_\mu(q^{\lambda+N+\rho}, q^{-\rho})$ is the Schur polynomial in N variables $\{q^{\lambda_i+i-\frac{1}{2}}\}_{1\leq i\leq N}$, which is a polynomial in q with non-negative integer coefficients and vanishes for $\ell(\lambda) \leq N < \ell(\mu)$. Therefore, when $Q = q^N$, we have

Proposition 4.4. *If* $N \in \mathbb{Z}_{\geq 0}$,

$$(-1)^{|\lambda|+|\mu|} \frac{Z_{\lambda,\mu}(q^N; q, q)}{Z_{\bullet,\bullet}(q^N; q, q)} = (-1)^{|\lambda|+|\mu|} f_\lambda f_\mu s_\mu\left(\{q^{\lambda_i+i-\frac{1}{2}}\}_{1\leq i\leq N}\right) s_\lambda\left(\{q^{i-\frac{1}{2}}\}_{1\leq i\leq N}\right), \tag{4.8}$$

which is a polynomial in q *with non-negative integer coefficients and vanishes for* $0 \leq N < \max\{\ell(\lambda), \ell(\mu)\}$. *This coincides with the colored Hopf link invariant by* [18] *up to the over all factor.*[9]

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[9]This was shown in [7] for $N \to \infty$.

Proof. If $0 \leq N < \ell(\lambda)$, $s_\lambda(q^{N+\rho}, q^{-\rho}) = 0$. If $\ell(\lambda) \leq N < \ell(\mu)$, $s_\mu(q^{\lambda+N+\rho}, q^{-\rho}) = 0$. By the symmetry $Z_{\lambda,\mu}(Q; q, q) = Z_{\mu,\lambda}(Q; q, q)$, we have $Z_{\lambda,\mu}(Q; q, q) = 0$ for $0 \leq N < \max\{\ell(\lambda), \ell(\mu)\}$. On the other hand, if $N \geq \max\{\ell(\lambda), \ell(\mu)\}$, $s_\mu(q^{\lambda+N+\rho}, q^{-\rho})$ is the Schur polynomial in N variables $\{q^{\lambda_i+i-\frac{1}{2}}\}_{1 \leq i \leq N}$, which is a polynomial in q with non-negative coefficients. $\square$

When $q \neq t$, it is difficult to calculate $Z_{\lambda,\mu}(Q; q, t)$ explicitly for lack of the cyclic symmetry of the refined topological vertex [7].

4.3 $\quad (\lambda, \mu) = (1^r, 1^s)$ case

For $\lambda = 1^r$ and $\mu = 1^s$, we have

Proposition 4.5.

$$\frac{Z_{1^r,1^s}(Q; q, t)}{Z_{\bullet,\bullet}(Q; q, t)} = g_{1^r} g_{1^s} e_r(Qt^\rho, t^{-\rho}) e_s(Qq^{(1^r)}t^\rho, t^{-\rho}), \tag{4.9}$$

which is a polynomial of degree $r + s$ in Q. Here $g_{1^r} := (-1)^r t^{-\frac{r(r-1)}{2}}$.

Proof. Since $s_{1^r}(x) = P_{1^r}(x; q, t) = e_r(x)$, it follows that

$$Z_{1^r,1^s}(Q; q, t) = \sum_\eta \frac{Q^{|\eta|}}{\langle P_\eta, P_\eta \rangle_{q,t}} P_\eta(t^{-\rho}; q, t) P_\eta(t^\rho; q, t) P_{1^r}(q^\eta t^\rho; q, t) e_s(q^\eta t^\rho). \tag{4.10}$$

The symmetry $P_\lambda(t^\rho; q, t) P_\mu(q^\lambda t^\rho; q, t) = P_\mu(t^\rho; q, t) P_\lambda(q^\mu t^\rho; q, t)$ in (A.11) leads to

$$Z_{1^r,1^s}(Q; q, t) = P_{1^r}(t^\rho; q, t) \sum_\eta \frac{Q^{|\eta|}}{\langle P_\eta, P_\eta \rangle_{q,t}} P_\eta(t^{-\rho}; q, t) P_\eta(q^{(1^r)}t^\rho; q, t) e_s(q^\eta t^\rho). \tag{4.11}$$

By (3.4), we can replace $e_s(q^\eta t^\rho)$ by H^s as follows:

$$Z_{1^r,1^s}(Q; q, t) = P_{1^r}(t^\rho; q, t) \sum_\eta \frac{1}{\langle P_\eta, P_\eta \rangle_{q,t}} P_\eta(Qq^{(1^r)}t^\rho; q, t) H^s P_\eta(x; q, t)|_{x=t^{-\rho}}. \tag{4.12}$$

The Cauchy formula (A.5) and (3.10) yield

$$\begin{aligned} Z_{1^r,1^s}(Q; q, t) &= P_{1^r}(t^\rho; q, t) H^s \Pi(x, Qq^{(1^r)}t^\rho; q, t)|_{x=t^{-\rho}} \\ &= g_{1^s} P_{1^r}(t^\rho; q, t) \Pi(t^{-\rho}, Qq^{(1^r)}t^\rho; q, t) e_s(Qq^{(1^r)}t^\rho, t^{-\rho}). \end{aligned} \tag{4.13}$$

But ((2.12) and (5.20) of [21])

$$\frac{\Pi(Qq^{(1^r)}t^\rho, t^{-\rho}; q, t)}{\Pi(Qt^\rho, t^{-\rho}; q, t)} = g_{1^r} \frac{P_{1^r}(Qt^\rho, t^{-\rho}; q, t)}{P_{1^r}(t^\rho; q, t)}, \tag{4.14}$$

which completes the proof. $\qquad\square$

Note that for $N \in \mathbb{Z}$ and $N \geq \ell(\lambda)$, $p_n(q^\lambda t^{N+\rho}, t^{-\rho}) = \sum_{i=1}^{N} q^{n\lambda_i} t^{n(N+\frac{1}{2}-i)}$. Thus, $e_r(q^\lambda t^{N+\rho}, t^{-\rho})$ is the elementary symmetric polynomial in N variables $\{q^{\lambda_i} t^{N+\frac{1}{2}-i}\}_{1\leq i\leq N}$, which is a polynomial in q and t with non-negative integer coefficients and vanishes for $\ell(\lambda) \leq N < r$. Therefore, when $Q = t^N$, we have

Proposition 4.6. *If $N \in \mathbb{Z}_{\geq 0}$,*

$$(-1)^{r+s} \frac{Z_{1^r,1^s}(t^N; q, t)}{Z_{\bullet,\bullet}(t^N; q, t)} = (-1)^{r+s} g_{1^s} g_{1^r} e_r(\{t^{i-\frac{1}{2}}\}_{1\leq i\leq N}) e_s(\{q^{(1^r)_i} t^{N+\frac{1}{2}-i}\}_{1\leq i\leq N}), \quad (4.15)$$

which is a polynomial in q and t with non-negative integer coefficients and vanishes for $0 \leq N < \max\{r, s\}$ (theorem 4.2 (iii), $\lambda = 1^r$ case).

When $s = 1$, up to an overall factor, (4.15) agrees to the recent result by Yonezawa [19] who used the method of matrix factorization developed by Khovanov and Rozansky [8] (theorem 4.2 (iv)).

In [20], based on an assumption on the computation of the refined topological vertex, Taki proposed that the homological invariants for the colored Hopf link are written as [10]

$$\bar{\mathcal{P}}_{\mu,\lambda}^{\text{Taki}}(v^{-1}Q; q^{-1}, t^{-1}) := \bar{c}_{\lambda,\mu} s_\lambda(t^\rho) s_\mu(q^\lambda t^\rho, \ Q^{-1}t^{-\rho}) N_{\bullet,\lambda}(Q^{-1}q/t; q, t) g_\lambda$$
$$= \bar{c}_{\lambda,\mu} s_\lambda(t^\rho) s_\mu(Qq^\lambda t^\rho, \ t^{-\rho}) N_{\lambda,\bullet}(Q; q, t),$$
$$\bar{c}_{\lambda,\mu} := (-1)^{|\lambda|} Q^{\frac{|\lambda|+|\mu|}{2}} v^{-2|\lambda||\mu|}. \quad (4.16)$$

with the denominator factor $N_{\lambda,\mu}$ of Nekrasov's partition function:

$$N_{\lambda,\mu}(Q; q, t) := \prod_{(i,j)\in\lambda} \left(1 - Q\, q^{\lambda_i-j} t^{\mu_j^\vee-i+1}\right) \prod_{(i,j)\in\mu} \left(1 - Q\, q^{-\mu_i+j-1} t^{-\lambda_j^\vee+i}\right). \quad (4.17)$$

For $(\lambda, \mu) = (1^r, 1^s)$,

$$\bar{\mathcal{P}}_{1^s,1^r}^{\text{Taki}}(v^{-1}Q; q^{-1}, t^{-1}) = \bar{c}_{1^r,1^s} s_{1^r}(t^\rho) e_s(Qq^{(1^r)} t^\rho, \ t^{-\rho}) N_{1^r,\bullet}(Q; q, t), \quad (4.18)$$

which coincides with (4.9) up to the over all factor $(-1)^{r+s+1} Q^{(r+s)/2} v^{-2rs}$.

4.4 $\ (\lambda, \mu) = (\lambda, 1^s)$ case

Since the set of the Macdonald functions $\{P_\lambda(x; q, t)\}_{|\lambda|=d}$ is a basis of the homogeneous symmetric functions of degree d, we can write the Schur function $s_\lambda(x)$ by the Macdonald functions as

$$s_\lambda(x) = \sum_{\mu\leq\lambda} U_{\lambda,\mu} P_\mu(x; q, t), \qquad U_{\lambda,\mu} := \sum_{\nu(\lambda\geq\nu\geq\mu)} u_{\lambda,\nu}(q, q)(u^{-1})_{\nu,\mu}(q, t), \quad (4.19)$$

---

[10]Our $(q, t) = (q^{-1}, t^{-1})$ of [20].

where $u_{\lambda,\mu}(q,t)$ is defined by (A.3). Note that $U_{\lambda,\mu}$ is a rational function in q and t. Then, we have

Proposition 4.7.

$$\frac{Z_{\lambda,1^s}(Q;q,t)}{Z_{\bullet,\bullet}(Q;q,t)} = g_{1^s} \sum_{\mu \le \lambda} U_{\lambda,\mu} g_\mu P_\mu(Qt^\rho, t^{-\rho}; q, t) e_s(Qq^\mu t^\rho, t^{-\rho}), \qquad (4.20)$$

which is a polynomial of degree $|\lambda| + s$ in Q. Here $g_\lambda := \prod_{(i,j) \in \lambda} (-1) q^{\lambda_i - j} t^{-\lambda_j^\vee + i}$.

Proof. First, we have

$$Z_{\lambda,1^s}(Q;q,t) = \sum_{\mu \le \lambda} U_{\lambda,\mu} \widetilde{Z}_{\mu,1^s}(Q;q,t), \qquad (4.21)$$

where

$$\widetilde{Z}_{\mu,1^s}(Q;q,t) := \sum_\eta \frac{Q^{|\eta|}}{\langle P_\eta, P_\eta \rangle_{q,t}} P_\eta(t^{-\rho}; q, t) P_\eta(t^\rho; q, t) P_\mu(q^\eta t^\rho; q, t) s_{1^s}(q^\eta t^\rho). \qquad (4.22)$$

Then, (A.11) leads to

$$\widetilde{Z}_{\mu,1^s}(Q;q,t) = \sum_\eta \frac{Q^{|\eta|}}{\langle P_\eta, P_\eta \rangle_{q,t}} P_\eta(t^{-\rho}; q, t) P_\mu(t^\rho; q, t) P_\eta(q^\mu t^\rho; q, t) s_{1^s}(q^\eta t^\rho). \qquad (4.23)$$

We can proceed in the same way as the last subsection. Namely, $s_{1^s}(x) = e_s(x)$, (3.4) and (3.10) yield

$$\begin{aligned}
\widetilde{Z}_{\mu,1^s}(Q;q,t) &= P_\mu(t^\rho; q, t) \sum_\eta \frac{Q^{|\eta|}}{\langle P_\eta, P_\eta \rangle_{q,t}} P_\eta(q^\mu t^\rho; q, t) H^s P_\eta(x; q, t)|_{x=t^{-\rho}}. \\
&= P_\mu(t^\rho; q, t) H^s \Pi(x, Qq^\mu t^\rho; q, t)|_{x=t^{-\rho}} \\
&= P_\mu(t^\rho; q, t) g_{1^s} e_s(Qq^\mu t^\rho, t^{-\rho}) \Pi(t^{-\rho}, Qq^\mu t^\rho; q, t). \qquad (4.24)
\end{aligned}$$

But from ((2.12) and (5.20) of [21])

$$\frac{\Pi(Qq^\mu t^\rho, t^{-\rho}; q, t)}{\Pi(Qt^\rho, t^{-\rho}; q, t)} = g_\mu \frac{P_\mu(Qt^\rho, t^{-\rho}; q, t)}{P_\mu(t^\rho; q, t)}, \qquad (4.25)$$

we conclude that

$$\frac{\widetilde{Z}_{\mu,1^s}(Q;q,t)}{Z_{\bullet,\bullet}(Q;q,t)} = g_{1^s} g_\mu P_\mu(Qt^\rho, t^{-\rho}; q, t) e_s(Qq^\mu t^\rho, t^{-\rho}). \qquad (4.26)$$

□

Note that for $N \in \mathbb{Z}$ and $N \ge \ell(\lambda)$, $P_\mu(q^\lambda t^{N+\rho}, t^{-\rho})$ is the Macdonald polynomial in N variables $\{q^{\lambda_i} t^{N+\frac{1}{2}-i}\}_{1 \le i \le N}$ and vanishes for $\ell(\lambda) \le N < \ell(\mu)$. Therefore, when

$Q = t^N$, we have

Proposition 4.8. *If $N \in \mathbb{Z}_{\geq 0}$,*

$$\frac{Z_{\lambda,1^s}(t^N; q, t)}{Z_{\bullet,\bullet}(t^N; q, t)} = g_{1^s} \sum_{\mu \leq \lambda} U_{\lambda,\mu} g_\mu P_\mu(\{t^{i-\frac{1}{2}}\}_{1 \leq i \leq N}; q, t) e_s(\{q^{\mu_i} t^{N+\frac{1}{2}-i}\}_{1 \leq i \leq N})$$

$$= g_{1^s} e_s(\{t^{i-\frac{1}{2}}\}_{1 \leq i \leq N}) \sum_{\mu \leq \lambda} U_{\lambda,\mu} g_\mu P_\mu(\{q^{(1^s)_i} t^{N+\frac{1}{2}-i}\}_{1 \leq i \leq N}; q, t), \quad (4.27)$$

which vanishes for $0 \leq N < \max\{\ell(\lambda), s\}$ (theorem 4.2 (ii), $\mu = 1^s$ case).

Finally, we should make a remark on the fact that the transition function $U_{\lambda,\mu}(q, t)$ in (4.19) is a rational function in q and t. It is not obvious that the formulas in the above propositions are in fact polynomials in q and t. However, we have checked the following conjecture up to $d = 7$ by direct calculation.

Conjecture 4.9. *For $|\lambda| = d$, $\sum_{\mu(\lambda \geq \mu \geq \nu)} U_{\lambda,\mu} g_\mu (U^{-1})_{\mu,\nu}$ is a polynomial with integer coefficients of degree $d(d-1)/2$ in q and of degree $d(d-1)/2$ in t^{-1}.*

On the assumption that the above conjecture is true, if $N \in \mathbb{Z}_{\geq 0}$ then

$$\frac{Z_{\lambda,1^s}(t^N; q, t)}{Z_{\bullet,\bullet}(t^N; q, t)} = g_{1^s} e_s(t^{N+\rho}, t^{-\rho}) \sum_{\substack{\mu,\nu \\ |\mu|=|\nu|=|\lambda|}} U_{\lambda,\mu} g_\mu (U^{-1})_{\mu,\nu} s_\nu(q^{(1^s)} t^{N+\rho}, t^{-\rho}) \quad (4.28)$$

is a polynomial in q and t with integer coefficients because $s_\lambda(x)$ is a function in x with non-negative integer coefficients (theorem 4.2 (iii), $|\lambda| \leq 7$ case).

4.5   General (λ, μ) case

From (A.1) and (A.3), we have

$$s_\lambda(x) = \sum_{\mu \leq \lambda} V_{\lambda,\mu} e_{\mu^\vee}(x), \qquad V_{\lambda,\mu} := \sum_{\nu(\lambda \geq \nu \geq \mu)} u_{\lambda,\nu}(a^{-1})_{\nu,\mu}, \quad (4.29)$$

where $u_{\lambda,\mu} := u_{\lambda,\mu}(q, q)$ and $a_{\lambda,\mu}$ is defined in (A.1). Note that $|\mu^\vee| = |\lambda|$ in the above equation. More precisely, we have the Jacobi-Trudy formula $s_\lambda(x) = \det \left(e_{\lambda_i^\vee - i + j}(x) \right)_{1 \leq i,j \leq \lambda_1}$ with $e_{-r} = 0$ for $r > 0$. Then, we have

Proposition 4.10.

$$\frac{Z_{\lambda,\mu}(Q; q, t)}{Z_{\bullet,\bullet}(Q; q, t)} = \sum_{\substack{\nu \leq \lambda \\ \sigma \leq \mu}} V_{\lambda,\nu} V_{\mu,\sigma} \left. \frac{H^{\nu^\vee}(x) H^{\sigma^\vee}(x) \Pi(x, Qt^{-\rho}; q, t)}{\Pi(x, Qt^{-\rho}; q, t)} \right|_{x = t^\rho}, \quad (4.30)$$

which is a polynomial of degree $|\lambda| + |\mu|$ in Q and vanishes when $Q = 1$ (theorem 4.2 (i)).

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Proof. (3.4) and (A.5) yield

$$Z_{\lambda,\mu}(Q;q,t) = \sum_{\eta} \frac{Q^{|\eta|}}{\langle P_\eta, P_\eta \rangle_{q,t}} P_\eta(t^{-\rho};q,t) \sum_{\substack{\nu \leq \lambda \\ \sigma \leq \mu}} V_{\lambda,\nu} V_{\mu,\sigma} H^{\nu^\vee}(x) H^{\sigma^\vee}(x) P_\eta(x;q,t) \Big|_{x=t^\rho}$$

$$= \sum_{\substack{\nu \leq \lambda \\ \sigma \leq \mu}} V_{\lambda,\nu} V_{\mu,\sigma} \, H^{\nu^\vee}(x) H^{\sigma^\vee}(x) \Pi\left(x, Qt^{-\rho};q,t\right) \Big|_{x=t^\rho}. \tag{4.31}$$

Then, the proposition in subsection 3.3 completes the proof. $\square$

Although $Z_{\lambda,\mu}(Q;q,t)$ is a power series in Q, we need its partial sum with degree $|\lambda| + |\mu|$ to calculate $Z_{\lambda,\mu}(Q;q,t)/Z_{\bullet,\bullet}(Q;q,t)$.

On the assumption that the conjecture 3.4 is true, if $Q = t^N$, $N \in \mathbb{Z}$ with $0 \leq N < \max\{\ell(\lambda), \ell(\mu)\}$, then (4.30) would vanish (theorem 4.2 (ii), $|\lambda| + |\mu| \leq 7$ case).

5 Towards a resolution of positivity problem

We now make a comment on the positivity problem of the GIKV's proposal. Let $(\mathbf{a}; \mathbf{q}, \mathbf{t}) := (Q^{-\frac{1}{2}}; t^{-\frac{1}{2}}, -(t/q)^{\frac{1}{2}})$. When $\mathbf{a} = \mathbf{q}^N$ with $N \in \mathbb{N}$, the superpolynomial of the homological invariants of the colored Hopf link reduces to $\sum_{i,j\in\mathbb{Z}} \mathbf{q}^i \mathbf{t}^j \dim \mathcal{H}_{i,j}^{\mathfrak{sl}(N);\lambda,\mu}$ with certain doubly graded homology $\mathcal{H}_{i,j}^{\mathfrak{sl}(N);\lambda,\mu}$ [7]. Therefore, it should be by definition a polynomial in $\mathbf{q}$ and $\mathbf{t}$ with *non-negative* integer coefficients. However, in general, $Z_{\lambda,\mu}^{\mathrm{inst}}(t^N;q,t)(-1)^{|\lambda|+|\mu|}$ is not so. For example,

$$-Z_{3,\bullet}^{\mathrm{inst}}(t^2;q,t) = q^3(t^6+t^5) + q^2(t^5-t^3) + q(t^5+t^4). \tag{5.1}$$

A solution to this positivity problem may be given by replacing the Schur function in (4.1) by the Macdonald function $P_\lambda(z;\tilde{q},\tilde{t})$ with $\tilde{t} = 0$ and appropriately chosen $\tilde{q}$. Let

$$\widetilde{Z}_{\lambda,\mu}(Q;q,t) := \sum_{\eta} (-v^{-1}Q)^{|\eta|} P_{\eta^\vee}(q^\rho;t,q) P_\eta(t^\rho;q,t) P_\lambda(q^\eta t^\rho;\tilde{q},0) P_\mu(q^\eta t^\rho;\tilde{q},0), \tag{5.2}$$

and $\widetilde{Z}_{\lambda,\mu}^{\mathrm{inst}}(Q;q,t) := \widetilde{Z}_{\lambda,\mu}(Q;q,t)/Z_{\bullet,\bullet}(Q;q,t)$. Note that $\widetilde{Z}_{\lambda,\mu}^{\mathrm{inst}}(Q;q,t)\Big|_{\tilde{q}=0} = Z_{\lambda,\mu}^{\mathrm{inst}}(Q;q,t)$ and $\widetilde{Z}_{1^r,1^s}^{\mathrm{inst}}(Q;q,t) = Z_{1^r,1^s}^{\mathrm{inst}}(Q;q,t)$, because of $P_\lambda(x;0,0) = s_\lambda(x)$ and $P_{1^r}(x;\tilde{q},\tilde{t}) = s_{1^r}(x)$, respectively. Since by (A.1) and (A.3),

$$P_\lambda(x;\tilde{q},\tilde{t}) = \sum_{\mu \leq \lambda} \widetilde{U}_{\lambda,\mu} P_\mu(x;q,t), \qquad \widetilde{U}_{\lambda,\mu} := \sum_{\nu(\lambda \geq \nu \geq \mu)} u_{\lambda,\nu}(\tilde{q},\tilde{t})(u^{-1})_{\nu,\mu}(q,t),$$

$$= \sum_{\mu \leq \lambda} \widetilde{V}_{\lambda,\mu} e_{\mu^\vee}(x), \qquad \widetilde{V}_{\lambda,\mu} := \sum_{\nu(\lambda \geq \nu \geq \mu)} u_{\lambda,\nu}(\tilde{q},\tilde{t})(a^{-1})_{\nu,\mu}, \tag{5.3}$$

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all propositions in subsections 4.3 and 4.4 hold with these $\widetilde{U}_{\lambda,\mu}$ and $\widetilde{V}_{\lambda,\mu}$.

If we choose the parameter $\tilde{q}$ appropriately, it may overcome the positivity problem. For example, when $\tilde{q} = q$,

$$-t^{\frac{3}{2}}\widetilde{Z}_{3,\bullet}^{\text{inst}}(t^2; q, t)\Big|_{\tilde{q}=q} = q^3(t^6 + t^5 + t^4 + t^3) + q(q+1)(t^5 + t^4). \tag{5.4}$$

More generally, for $Q = t^N$,

$$- t^{\frac{3}{2}}\widetilde{Z}_{3,\bullet}^{\text{inst}}(t^N; q, t) = \tilde{q}^3 t^3 \begin{bmatrix} N \\ 3 \end{bmatrix}_t + \tilde{q}(\tilde{q}+1)\left(qQt\begin{bmatrix} N \\ 2 \end{bmatrix}_t + t^4\begin{bmatrix} N \\ 3 \end{bmatrix}_t\right)[2]_t$$
$$+ q^3 Q^2 t[N]_t + q^2 Qt(t^2 - 1)\begin{bmatrix} N \\ 2 \end{bmatrix}_t + qQt^2\begin{bmatrix} N \\ 2 \end{bmatrix}_t [2]_t + t^6\begin{bmatrix} N \\ 3 \end{bmatrix}_t. \tag{5.5}$$

But if we choose $\tilde{q} = q$, then the negative coefficient in the q^2-term vanishes as

$$-t^{\frac{3}{2}}\widetilde{Z}_{3,\bullet}^{\text{inst}}(t^N; q, t)\Big|_{\tilde{q}=q} = q^3\left(Q^2[N]_t + Q\begin{bmatrix} N \\ 2 \end{bmatrix}_t [2]_t + t^2\begin{bmatrix} N \\ 3 \end{bmatrix}_t\right)t$$
$$+ q(q+1)\left(Q\begin{bmatrix} N \\ 2 \end{bmatrix}_t + t^2\begin{bmatrix} N \\ 3 \end{bmatrix}_t\right)t^2[2]_t + t^6\begin{bmatrix} N \\ 3 \end{bmatrix}_t$$
$$= q^3\begin{bmatrix} N+2 \\ 3 \end{bmatrix}_t + q(q+1)t[2]_t\begin{bmatrix} N+1 \\ 3 \end{bmatrix}_t + t^3\begin{bmatrix} N \\ 3 \end{bmatrix}_t. \tag{5.6}$$

Note that the q-integer $[N]_t := \frac{1-t^N}{1-t}$ and the q-binomial coefficient $\begin{bmatrix} N \\ r \end{bmatrix}_t := \prod_{i=1}^r \frac{1-t^{N-r+i}}{1-t^i}$ are polynomials in t with non-negative integer coefficients for $N, r \in \mathbb{N}$. We checked that $(-1)^{|\lambda|+s}\widetilde{Z}_{\lambda,1^s}^{\text{inst}}(t^N; q, t)|_{\tilde{q}=q}$ ($N \in \mathbb{Z}_{\geq 0}$) is a polynomial in q and t with non-negative integer coefficients for $|\lambda| + s \leq 5$ (see appendix F).

For non-antisymmetric representations, the specialization $\tilde{q} = q$ fails to solve the positivity problem. For example,

$$\widetilde{Z}_{2,2}^{\text{inst}}(t^2; q, t)/(qt)^2\Big|_{\tilde{q}=q} = q^4 t^5[2]_t + q^3 t^3(t^2 - 1) + q^2 t^3(t^2 + 3t + 1) + qt^3(t^2 + 2t + 3) + t^2[3]_t. \tag{5.7}$$

However, if we choose $\tilde{q} = (1 + qt)q/t + p$ with arbitrary p, then

$$\widetilde{Z}_{2,2}^{\text{inst}}(t^2; q, t)/(qt)^2\Big|_{\tilde{q}=(1+qt)q/t+p} = p^2 t^2 + 2p\{q^2 t^2 + qt(t^3 + t^2 + 1) + t^3\}$$
$$+ q^4 t^2(t^4 + t^3 + 1) + q^3 t(t^4 + 2t^3 + t^2 + 2) + q^2(t^5 + t^4 + 3t^3 + t^2 + 1)$$
$$+ qt^2(t^3 + 2t^2 + t + 2) + t^2[3]_t. \tag{5.8}$$

We checked that $(-1)^{|\lambda|+|\mu|}\widetilde{Z}_{\lambda,\mu}^{\text{inst}}(t^N; q, t)|_{\tilde{q}=(1+qt)q/t+p}$ ($N \in \mathbb{Z}_{\geq 0}$) is a polynomial in q, t and p with non-negative integer coefficients for $|\lambda|+|\mu| \leq 5$ (see appendix F). Although an interpretation from the viewpoint of homological algebra is unclear, $\widetilde{Z}_{\lambda,\mu}^{\text{inst}}(Q; q, t)|_{\tilde{q}=(1+qt)q/t+p}$ gives a generalization of the colored Hopf link invariants with a third parameter p in the sense that it reduces to $Z_{\lambda,\mu}^{\text{inst}}(Q; q, t)$ of GIKV when $\tilde{q} = 0$, i.e. $p = -(1 + qt)q/t$.

Acknowledgments

Appendix A. Symmetric functions

Here we recapitulate basic properties of the symmetric functions in $x = (x_1, x_2, \cdots)$ [22]. The monomial symmetric function is defined by $m_\lambda := \sum_\sigma x_1^{\lambda_{\sigma(1)}} x_2^{\lambda_{\sigma(2)}} \cdots$, where the summation is over all distinct permutations of $(\lambda_1, \lambda_2, \cdots)$. The power sum symmetric function is $p_n := \sum_{i=1}^\infty x_i^n$. The elementary symmetric function defined by

$$e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots, \qquad \sum_{r \geq 0} w^r e_r := \prod_{i \geq 1} (1 + x_i w) = \exp \left\{ - \sum_{n > 0} \frac{(-w)^n}{n} p_n \right\}$$

enjoys

$$e_{\lambda^\vee} = \sum_{\mu \leq \lambda} a_{\lambda\mu} m_\mu, \quad a_{\lambda\lambda} = 1, \quad a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}. \tag{A.1}$$

For any symmetric functions f and g, in power sums p_n's, we define a scalar product as

$$\langle f(p), g(p) \rangle_{q,t} := f(p^*) \, g(p) \,|_{\text{constant part}} p_n^* := n \frac{1 - q^n}{1 - t^n} \frac{\partial}{\partial p_n}.$$

The Macdonald symmetric function $P_\lambda(x; q, t)$ is uniquely specified by the following orthogonality and normalization:

$$\langle P_\lambda(x; q, t), P_\mu(x; q, t) \rangle_{q,t} = 0 \qquad \text{if } \lambda \neq \mu, \tag{A.2}$$

$$P_\lambda(x; q, t) = \sum_{\mu \leq \lambda} u_{\lambda\mu}(q, t) m_\mu(x), \quad u_{\lambda\lambda}(q, t) = 1, \quad u_{\lambda\mu}(q, t) \in \mathbb{Q}(q, t). \tag{A.3}$$

Here we used the dominance partial ordering on the Young diagrams defined as $\lambda \geq \mu \Leftrightarrow |\lambda| = |\mu|$ and $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i. Note that $P(x; q^{-1}, t^{-1}) = P(x; q, t)$. The scalar product is given by

$$\langle P_\lambda(x; q, t), P_\lambda(x; q, t) \rangle_{q,t} = \prod_{(i,j) \in \lambda} \frac{1 - q^{\lambda_i - j + 1} t^{\lambda_j^\vee - i}}{1 - q^{\lambda_i - j} t^{\lambda_j^\vee - i + 1}}.$$

We abbreviate it to $\langle P_\lambda, P_\lambda \rangle_{q,t}$. The skew-Macdonald symmetric function $P_{\lambda/\mu}(x; q, t)$ is defined by

$$P_{\lambda/\mu}(x; q, t) := P_\mu^*(x; q, t) \, P_\lambda(x; q, t) / \langle P_\mu, P_\mu \rangle_{q,t}.$$

Let $x = (x_1, x_2, \cdots)$ and $y = (y_1, y_2, \cdots)$ be two sets of variables. Then, we have

$$\sum_\mu P_{\lambda/\mu}(x; q, t) P_{\mu/\nu}(y; q, t) = P_{\lambda/\nu}(x, y; q, t), \tag{A.4}$$

where $P_{\lambda/\nu}(x, y; q, t)$ denotes the skew-Macdonald function in the set of variables $(x_1, x_2, \cdots, y_1, y_2, \cdots)$. The following Cauchy formula is especially important:

$$\sum_\lambda \frac{\langle P_\mu, P_\mu \rangle_{q,t}}{\langle P_\lambda, P_\lambda \rangle_{q,t}} P_{\lambda/\mu}(x; q, t) P_\lambda(y; q, t) = \Pi(x, y; q, t) P_\mu(y; q, t) \tag{A.5}$$

with

$$\Pi(x, y; q, t) := \exp \left\{ \sum_{n>0} \frac{1}{n} \frac{1 - t^n}{1 - q^n} p_n(x) p_n(y) \right\}.$$

We denote

$$p_n(q^\lambda t^\rho) := \sum_{i=1}^{\ell(\lambda)} (q^{n\lambda_i} - 1) t^{n(\frac{1}{2} - i)} + \frac{1}{t^{\frac{n}{2}} - t^{-\frac{n}{2}}} = \sum_{i=1}^{N} q^{n\lambda_i} t^{n(\frac{1}{2} - i)} + \frac{t^{-nN}}{t^{\frac{n}{2}} - t^{-\frac{n}{2}}} \tag{A.6}$$

for any $N \geq \ell(\lambda)$. Let $p_n(x, y) := p_n(x) + p_n(y)$; then,

$$p_n(cq^\lambda t^\rho, Lt^{-\rho}) = c^n \sum_{i=1}^{\ell(\lambda)} (q^{n\lambda_i} - 1) t^{n(\frac{1}{2} - i)} + \frac{c^n - L^n}{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}, \qquad c, L \in \mathbb{C}. \tag{A.7}$$

Note that [10]

$$P_{\mu^\vee}(-t^{\lambda^\vee} q^\rho; t, q) = \frac{v^{|\mu|}}{\langle P_\mu, P_\mu \rangle_{q,t}} P_\mu(q^{-\lambda} t^{-\rho}; q, t). \tag{A.8}$$

The Macdonald function in the power sums $p_n = (1 - L^n)/(t^{\frac{n}{2}} - t^{-\frac{n}{2}})$ is [22](Ch. VI.6)

$$P_\lambda(t^\rho, Lt^{-\rho}; q, t) = \prod_{(i,j) \in \lambda} (-1) t^{\frac{1}{2}} q^{j-1} \frac{1 - Lq^{1-j} t^{i-1}}{1 - q^{\lambda_i - j} t^{\lambda_j^\vee - i + 1}} \tag{A.9}$$

for a generic $L \in \mathbb{C}$. Note that

$$P_\lambda(t^\rho; q, t) P_{\lambda^\vee}(q^\rho; t, q) = P_\lambda(t^{-\rho}; q, t) P_{\lambda^\vee}(q^{-\rho}; t, q)$$
$$= \prod_{(i,j) \in \lambda} \frac{(q/t)^{\frac{1}{2}}}{(1 - q^{-\lambda_i + j} t^{-\lambda_j^\vee + i - 1})(1 - q^{\lambda_i - j + 1} t^{\lambda_j^\vee - i})}. \tag{A.10}$$

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If $L = t^{-N}$ with $N \in \mathbb{N}$ and $N \geq \ell(\lambda)$, then

$$p_n(q^\lambda t^\rho, t^{-N-\rho}) = \sum_{i=1}^{N} q^{n\lambda_i} t^{n(\frac{1}{2}-i)}$$

is the power sum symmetric polynomial in N variables $\{q^{\lambda_i} t^{\frac{1}{2}-i}\}_{1 \leq i \leq N}$; hence, $P_\lambda(t^\rho, t^{-N-\rho}; q, t)$ reduces to the Macdonald symmetric polynomial in N variables. Therefore,

$$P_\lambda(t^\rho, t^{-N-\rho}; q, t) = 0 \quad \text{for} \quad \ell(\lambda) > N \in \mathbb{N}.$$

For $N \in \mathbb{N}$, there is a symmetry [22](Ch. VI.6)

$$P_\lambda(t^\rho, t^{-N-\rho}; q, t) P_\mu(q^\lambda t^\rho, t^{-N-\rho}; q, t) = P_\mu(t^\rho, t^{-N-\rho}; q, t) P_\lambda(q^\mu t^\rho, t^{-N-\rho}; q, t). \quad \text{(A.11)}$$

The Hall-Littlewood and Schur functions are defined by $P_\lambda(x; t) := P_\lambda(x; 0, t)$ and $s_\lambda(x) := P_\lambda(x; q, q)$, respectively. Note that $p_n(x)$, $s_\lambda(x)$ and $e_r(x) = P_{1^r}(x; q, t)$ are symmetric functions in x with non-negative integer coefficients. Note that

$$\sum_{s=0}^{r} e_{r-s}(x) e_s(y) = e_r(x, y).$$

For $\lambda = 1^r$, (A.9) reduces to

$$e_r(t^\rho, Lt^{-\rho}) = \prod_{i=1}^{r} (-1) t^{\frac{1}{2}} \frac{1 - Lt^{i-1}}{1 - t^i}. \quad \text{(A.12)}$$

Note also that

$$P_\lambda(t^\rho; t) = \delta_{\lambda, 1^r} e_r(t^\rho).$$

The q-integer and the q-binomial coefficient

$$[N]_t := \frac{1 - t^N}{1 - t}, \qquad \begin{bmatrix} N \\ r \end{bmatrix}_t := \prod_{i=1}^{r} \frac{1 - t^{N-r+i}}{1 - t^i}$$

are polynomials in t with non-negative integer coefficients for $N, r \in \mathbb{N}$. Note that

$$t^{-\frac{n}{2}} p_n(t^{N+\rho}, t^{-\rho}) = \frac{1 - t^{nN}}{1 - t^n} = [N]_{t^n}, \qquad t^{-\frac{r}{2}} e_r(t^{N+\rho}, t^{-\rho}) = t^{\frac{r(r-1)}{2}} \begin{bmatrix} N \\ r \end{bmatrix}_t.$$

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Appendix B. Proof of (2.5)

Here we prove (2.5). For $\sigma \in \mathcal{S}_N$, let $d(\sigma) := \#\{ (i,j) \mid i < j, \ \sigma(i) > \sigma(j) \}$ be the inversion number and let

$$\Delta_\sigma := t^{d(\sigma)} \exp\left\{ \sum_{n>0} \frac{t^n - t^{-n}}{n} \sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} \frac{x_{\sigma(i)}^n}{x_{\sigma(j)}^n} \right\}. \tag{B.1}$$

Then,

$$\Delta(\bar{x};t)^{-1}\Delta_\sigma = t^{d(\sigma)} \exp\left\{ \sum_{n>0} \frac{1-t^n}{n} \left(\sum_{\substack{i<j \\ \sigma(i)<\sigma(j)}} \frac{x_{\sigma(j)}^n}{x_{\sigma(i)}^n} - t^{-n} \sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} \frac{x_{\sigma(i)}^n}{x_{\sigma(j)}^n} \right) \right\}, \tag{B.2}$$

which is a formal power series in $\{x_j/x_i\}_{i<j}$ and is equivalent to $\sigma\Delta(\bar{x};t)^{-1} = \prod_{i<j} \frac{1-tx_{\sigma(j)}/x_{\sigma(i)}}{1-x_{\sigma(j)}/x_{\sigma(i)}}$ by the analytic continuation. Since $P_\lambda(x;t)$ in (2.1) is a polynomial in x, it follows that

$$P_\lambda(x;t) = v_\lambda^{-1}(t)\Delta(\bar{x};t)^{-1} \sum_{\sigma\in\mathcal{S}_N} \Delta_\sigma \prod_{i=1}^N x_{\sigma(i)}^{\lambda_i} \tag{B.3}$$

and

$$v_\lambda(t)P_\lambda(x;t)\Delta(\bar{x};t) = \sum_{\nu\geq\lambda}\sum_{\sigma\in\mathcal{S}_N} u_{\lambda,\nu}^\sigma \prod_{i=1}^N x_{\sigma(i)}^{\nu_i}, \qquad u_{\lambda,\lambda}^\sigma = t^{d(\sigma)}. \tag{B.4}$$

Here ν is a sequence of N integers $\nu = (\nu_1, \nu_2, \cdots, \nu_N)$, $\nu_i \in \mathbb{Z}$ and $\geq$ is the dominance partial ordering defined as $\nu \geq \lambda \Leftrightarrow \sum_i \nu_i = \sum_i \lambda_i$ and $\nu_1 + \cdots + \nu_k \geq \lambda_1 + \cdots + \lambda_k$ for all k. Thus, we have

$$\begin{aligned}
v_\lambda(t)\langle P_\lambda(x;t), x^\lambda \rangle''_{N;t} &= 1, \\
v_\lambda(t)\langle P_\lambda(x;t), \prod_i x_{\sigma(i)}^{\lambda_i} \rangle''_{N;t} &= t^{d(\sigma)}, \\
v_\lambda(t)\langle P_\lambda(x;t), \prod_i x_{\sigma(i)}^{\mu_i} \rangle''_{N;t} &= 0, \qquad \mu < \lambda.
\end{aligned} \tag{B.5}$$

Therefore, by (A.3) we conclude that

$$v_\lambda(t)\langle P_\lambda(x;t), P_\mu(x;t) \rangle''_{N;t} = \delta_{\lambda,\mu}[N]!_t. \tag{B.6}$$

Here we use the identity $\sum_{\sigma\in\mathcal{S}_N} t^{d(\sigma)} = [N]!_t$, which is proved by the induction in N. This completes the proof of (2.5).

Appendix C. Macdonald operators

Here we define (higher order) Macdonald operators which are compatible with tending the number of variables to infinity [16][23][14]. For each integer r such that $0 \leq r \leq N$, let D_N^r be the Macdonald operators in N variables $x = (x_1, \cdots, x_N)$, $D_N^0 := 1$ and

$$D_N^r := t^{r(r-1)/2} \sum_{\substack{I \\ \sharp I = r}} \prod_{\substack{i \in I \\ j \notin I}} \frac{t x_i - x_j}{x_i - x_j} \prod_{i \in I} T_{q, x_i}, \tag{C.1}$$

summed over all r-element subsets I of $\{1, 2, \cdots, N\}$. We set $D_N^r := 0$, $r > N$. Here $T_{q,x}$ is the q-shift operator such that $T_{q,x} f(x) = f(qx)$. Let $D_N(\tilde{w}) := \sum_{r=0}^{N} D_N^r \tilde{w}^r$; then, the Macdonald polynomial is the eigenfunction for D_N:

$$D_N(\tilde{w}) P_\lambda(x; q, t) = P_\lambda(x; q, t) \varepsilon_{N, \lambda},$$
$$\varepsilon_{N, \lambda} := \prod_{i=1}^{N} (1 + \tilde{w} q^{\lambda_i} t^{N-i}) = \sum_{r=0}^{N} \tilde{w}^r e_r(q^\lambda t^{N-\frac{1}{2}+\rho}, t^{\frac{1}{2}-\rho}). \tag{C.2}$$

Therefore, D_N^r are simultaneously diagonalized by the Macdonald polynomials

$$D_N^r P_\lambda(x; q, t) = P_\lambda(x; q, t) e_r(q^\lambda t^{N-\frac{1}{2}+\rho}, t^{\frac{1}{2}-\rho}); \tag{C.3}$$

thus, D_N^r commute with each other $[D_N^r, D_N^s] = 0$ on the space of the symmetric function in N variables. Note that D_N^r is not compatible with the restriction of the variables defined by setting $x_N = 0$:

$$D_N^r|_{x_N=0} = t^r D_{N-1}^r + t^{r-1} D_{N-1}^{r-1},$$
$$D_N(\tilde{w})|_{x_N=0} = (1 + \tilde{w}) D_{N-1}(t\tilde{w}). \tag{C.4}$$

So we need to modify it to take $N \to \infty$. By using $p_n(t^{\rho-\frac{1}{2}}) = 1/(t^n - 1)$ and $\exp\left\{\sum_{n>0} \frac{1}{n} \frac{(-\tilde{w})^n}{1-t^n}\right\} = \sum_{m \geq 0} t^{-\frac{m}{2}} \tilde{w}^m e_m(t^\rho)$, let

$$H_N := D_N \exp\left\{\sum_{n>0} \frac{1}{n} \frac{(-\tilde{w})^n}{1-t^n}\right\} =: \sum_{r \geq 0} w^r H_N^r, \qquad w := \tilde{w} t^{N-\frac{1}{2}},$$

$$H_N^r = \sum_{s=0}^{\min(r, N)} t^{\frac{s}{2}-rN} e_{r-s}(t^\rho) D_N^s, \qquad r = 0, 1, 2, \cdots. \tag{C.5}$$

Then, by (A.6), we obtain

$$E_{N, \lambda} := \exp\left\{\sum_{n>0} \frac{1}{n} \frac{(-\tilde{w})^n}{1-t^n}\right\} \varepsilon_{N, \lambda} = \sum_{r \geq 0} w^r e_r(q^\lambda t^\rho), \tag{C.6}$$

which is independent of N for any $N \geq \ell(\lambda)$. Thus,

$$
\begin{aligned}
H_N P_\lambda(x; q, t) &= P_\lambda(x; q, t) E_{N,\lambda}, \\
H_N^r P_\lambda(x; q, t) &= P_\lambda(x; q, t) e_r(q^\lambda t^\rho).
\end{aligned}
\tag{C.7}
$$

Appendix D. Proof of (3.4)

Here we prove (3.4) by comparing H^r with H_N^r in (C.5). In this subsection, we suppose that the number of variables $x = (x_1, \cdots, x_N)$ is finite by setting $x_i = 0$, $i \geq N + 1$ and $p_n(x) = \sum_{i=1}^N x_i^n$, $n \in \mathbb{N}$. For each integer r, we denote $\widetilde{H}_N^r := t^{rN} H^r / e_r(t^\rho)$,

$$
\widetilde{H}_N^r = \oint \prod_{\alpha=1}^r \frac{dz_\alpha}{2\pi i z_\alpha} \prod_{\alpha=1}^r \prod_{j=1}^N \frac{t - x_j z_\alpha}{1 - x_j z_\alpha} \prod_{\alpha < \beta} \frac{1 - z_\alpha/z_\beta}{1 - z_\alpha/z_\beta t} \exp \left\{ \sum_{n>0} (q^n - 1) \sum_{\alpha=1}^r z_\alpha^{-n} \frac{\partial}{\partial p_n} \right\}.
\tag{D.1}
$$

Here z_α and p_n are formal parameters. For abbreviation, we write $1/(1 - z)$ instead of $\sum_{n \geq 0} z^n$ for the formal parameter z. Note that the operators $\widetilde{H}_N^r$ are compatible with the restriction of the variables defined by setting $x_N = 0$, $\widetilde{H}_{N-1}^r = t^{-r} \widetilde{H}_N^r|_{x_N=0}$. We also denote $\widetilde{H}_{N-1,(i)}^r := t^{-r} \widetilde{H}_N^r|_{x_i=0}$.

Instead of the rational function $(tx_i - x_j)/(x_i - x_j)$, we use

$$
\begin{aligned}
[i, j] &:= \frac{1 - tx_i/x_j}{1 - x_i/x_j} := (1 - tx_i/x_j) \sum_{n \geq 0} (x_i/x_j)^n, \\
[j, i] &:= \frac{t - x_i/x_j}{1 - x_i/x_j} := (t - x_i/x_j) \sum_{n \geq 0} (x_i/x_j)^n, \qquad i > j.
\end{aligned}
\tag{D.2}
$$

Then, we have the following recurrence relation.

Lemma D.1. *Suppose $x_i \neq x_j$ if $i \neq j$. For any $N \in \mathbb{Z}_{\geq 0}$ and $r \in \mathbb{N}$,*

$$
\widetilde{H}_N^r = \widetilde{H}_N^{r-1} + t^{r-1}(t-1) \sum_{\substack{i=1 \\ (x_i \neq 0)}}^N \widetilde{H}_{N-1,(i)}^{r-1} \prod_{j(\neq i)} [i, j] T_{q,x_i}.
\tag{D.3}
$$

Proof. For $p_n = \sum_i x_i^n$, $n \in \mathbb{N}$, since $T_{q,x_i} p_n = ((q^n - 1)x_i^n + p_n)$, we have for any function f in p

$$
T_{q,x_i} f(p(x)) = \exp \left\{ \sum_{N>0} (q^n - 1) z^{-n} \frac{\partial}{\partial p_n} \right\} f(p(x))|_{z=x_i^{-1}}.
\tag{D.4}
$$

The constant term in z_r is represented as the contour integral surrounding ∞ and is written by the summation of the residues at $z_r = \infty$ and $1/x_i$ with $x_i \neq 0$ as follows [Fig. 1]:
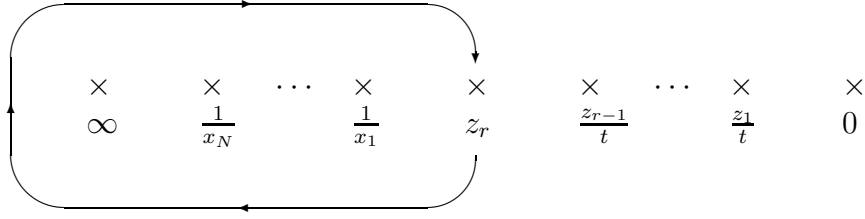
Figure 1: The constant term in z_r can be represented as the contour integral.

$$\widetilde{H}_N^r = \oint \prod_{\alpha=1}^{r-1} \frac{dz_\alpha}{2\pi i z_\alpha} \prod_{\alpha=1}^{r-1} \prod_{j=1}^{N} \frac{t - x_j z_\alpha}{1 - x_j z_\alpha} \prod_{\alpha<\beta}^{r-1} \frac{1 - z_\alpha/z_\beta}{1 - z_\alpha/z_\beta t} \exp\left\{ \sum_{n>0} (q^n - 1) \sum_{\alpha=1}^{r-1} z_\alpha^{-n} \frac{\partial}{\partial p_n} \right\}$$

$$+ (t-1) \sum_{i=1}^{N} \oint \prod_{\alpha=1}^{r-1} \frac{dz_\alpha}{2\pi i z_\alpha} \prod_{j(\neq i)} [i,j] \prod_{\alpha=1}^{r-1} \frac{1 - z_\alpha x_i}{1 - z_\alpha x_i/t} T_{q,x_i}$$

$$\times \prod_{\alpha=1}^{r-1} \prod_{j=1}^{N} \frac{t - x_j z_\alpha}{1 - x_j z_\alpha} \prod_{\alpha<\beta}^{r-1} \frac{1 - z_\alpha/z_\beta}{1 - z_\alpha/z_\beta t} \exp\left\{ \sum_{n>0} (q^n - 1) \sum_{\alpha=1}^{r-1} z_\alpha^{-n} \frac{\partial}{\partial p_n} \right\}$$

$$= \widetilde{H}^{r-1} + t^{r-1}(t-1) \sum_{i=1}^{N} \oint \prod_{\alpha=1}^{r-1} \frac{dz_\alpha}{2\pi i z_\alpha} \prod_{j(\neq i)} [i,j] \cdot T_{q,x_i}$$

$$\times \prod_{\alpha=1}^{r-1} \prod_{j(\neq i)} \frac{t - x_j z_\alpha}{1 - x_j z_\alpha} \prod_{\alpha<\beta}^{r-1} \frac{1 - z_\alpha/z_\beta}{1 - z_\alpha/z_\beta t} \exp\left\{ \sum_{n>0} (q^n - 1) \sum_{\alpha=1}^{r-1} z_\alpha^{-n} \frac{\partial}{\partial p_n} \right\}. \quad \text{(D.5)}$$

$\square$

Let us denote $I_i \oplus \{i\} \oplus J_i := \{1, 2, \cdots, N\}$. Let

$$D_{N-1,(k)}^r := t^{r(r-1)/2} \sum_{\substack{I_k \\ \sharp I_k = r}} \prod_{\substack{i \in I_k \\ j \in J_k}} [i,j] \prod_{i \in I_k} T_{q,x_i}; \quad \text{(D.6)}$$

then, we have the following recurrence relation.

Lemma D.2. *For any* $r = 0, 1, \cdots, N-1$,

$$\sum_{i=1}^{N} \prod_{j(\neq i)} [i,j] \cdot D_{N-1,(i)}^r T_{q,x_i} = t^{-r} \frac{t^{r+1} - 1}{t - 1} D_N^{r+1}. \quad \text{(D.7)}$$

Proof.

$$t^{-r(r-1)/2} \times \text{lhs} = \sum_{i=1}^{N} \prod_{j(\neq i)} [i,j] \cdot T_{q,x_i} \sum_{\substack{I_i \\ \sharp I_i = r}} \prod_{\substack{k \in I_i \\ \ell \in J_i}} [k,\ell] \prod_{k \in I_i} T_{q,x_k}$$

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$$
\begin{aligned}
&= \sum_{i=1}^{N} \sum_{I_i} \prod_{j \in I_i \cup J_i} [\,i,j\,] \cdot T_{q,x_i} \prod_{\substack{k \in I_i \\ \ell \in J_i}} [\,k,\ell\,] \prod_{k \in I_i} T_{q,x_k} \\
&= \sum_{i=1}^{N} \sum_{I_i} \prod_{j \in I_i} [\,i,j\,] \prod_{\substack{k \in I \\ \ell \in J_i}} [\,k,\ell\,] \prod_{k \in I} T_{q,x_k}, \qquad I := I_i \oplus \{i\}, \\
&= \sum_{\substack{I \\ \sharp I = r+1}} \sum_{\substack{i \in I}} \prod_{\substack{j \in I \\ j \neq i}} [\,i,j\,] \prod_{\substack{k \in I \\ \ell \notin I}} [\,k,\ell\,] \prod_{k \in I} T_{q,x_k}. \tag{D.8}
\end{aligned}
$$

Thus, it is sufficient to show that $\sum_{i=1}^{r+1} \prod_{j(\neq i)} [\,i,j\,] = \sum_{i=0}^{r} t^i$, which is proved as follows.
(i) Since the residues at $x_i = x_j$ vanish, so the lhs is a constant.
(ii) By putting $x_i = \epsilon^i$ and taking $\epsilon \to \infty$, we obtain the rhs. $\qquad\square$

Hence, we have

Lemma D.3.

$$
\widetilde{H}_N^r = \sum_{k=0}^{r} D_N^k \prod_{i=0}^{k-1} (t^{r-i} - 1). \tag{D.9}
$$

Proof. We proceed by induction on r. When $r = 0$, since $\widetilde{H}_N^0 = 1$, (D.9) holds. So assume that the result is true for $r - 1 \geq 0$. From (D.3) we obtain

$$
\begin{aligned}
\widetilde{H}_N^r &= \sum_{k=0}^{r-1} \prod_{s=0}^{k-1} (t^{r-s-1} - 1) \left(D_N^k + t^{r-1}(t-1) \sum_{i=1}^{N} D_{N-1,(i)}^k \prod_{j(\neq i)} [\,i,j\,] T_{q,x_i} \right) \\
&= \sum_{k=0}^{r-1} \prod_{s=0}^{k-1} (t^{r-s-1} - 1) \left(D_N^k + (t^r - t^{r-k-1}) D_N^{k+1} \right) \\
&= \left(\sum_{k=0}^{r-1} (t^{r-k} - 1) + \sum_{k=1}^{r} (t^r - t^{r-k}) \right) D_N^k \prod_{s=1}^{k-1} (t^{r-s} - 1). \tag{D.10}
\end{aligned}
$$

$\qquad\square$

Therefore,

Proposition D.4.

$$
H^r = \sum_{s=0}^{\min(r,N)} t^{\frac{s}{2} - rN} e_{r-s}(t^\rho) D_N^s, \qquad r = 0, 1, 2, \cdots. \tag{D.11}
$$

This completes the proof of (3.4) by taking the limit $N \to \infty$.

Since the Macdonald functions for all partitions form a basis of the ring of symmetric functions, H^r commute with each other on the space of symmetric functions.

Appendix E. Torus knot

Not only for the Hopf link but the homological invariants for other link may be related with the refined topological vertex or the Macdonald polynomial. For example, in [12], a reduced polynomial for the torus knot $T_{m,n}$ is conjectured as

$$\mathcal{P}(T_n) := \lim_{m \to \infty} \mathcal{P}(T_{m,n}) = \frac{1 - a^2 \mathbf{t} u}{1 - \mathbf{t}^{-2} u^2} \frac{1 - a^2 \mathbf{t} u^2}{1 - \mathbf{t}^{-2} u^3} \cdots \frac{1 - a^2 \mathbf{t} u^{n-1}}{1 - \mathbf{t}^{-2} u^n} \tag{E.1}$$

with $u := (\mathbf{qt})^2$. But this is nothing but the following specialization of the Macdonald polynomial:

$$P_{(n-1)}(Qt^{\frac{1}{2}+\rho}, t^{-\frac{1}{2}-\rho}; q, t) = \prod_{i=0}^{n-2} \frac{1 - q^i t Q}{1 - q^i t} \tag{E.2}$$

with $(Q; q, t) = (a^2 \mathbf{t}/\mathbf{q}^2; \mathbf{q}^2 \mathbf{t}^2, \mathbf{q}^4 \mathbf{t}^2)$. Note that $p_n(Qt^{\frac{1}{2}+\rho}, t^{-\frac{1}{2}-\rho}) = (1 - (Qt)^n)/(1 - t^n)$.

Appendix F. Example for $\widetilde{Z}_{\lambda,\mu}^{\mathrm{inst}}(t^N; q, t)$

Here we list the example for $\widetilde{Z}_{\lambda,\mu}^{\mathrm{inst}}(t^N; q, t)$ in section 5 with $Q = t^N$, which is written as a linear combination of the q-binomial coefficients $\begin{bmatrix} N \\ r \end{bmatrix}_t$ with $r \leq |\lambda| + |\mu|$. When $\tilde{q} = (1 + qt)q/t + p$ for $|\lambda| + |\mu| \leq 5$, we checked that any coefficient of them is a polynomial in q, t and p with non-negative integer coefficients, so is $\widetilde{Z}_{\lambda,\mu}^{\mathrm{inst}}(t^N; q, t)$. The same is true when $\tilde{q} = q$, i.e. $p = q - (1 + qt)q/t$, for $(\lambda, \mu) = (\lambda, 1^s)$ with $|\lambda| + s \leq 5$.

$$-t^{\frac{1}{2}} \widetilde{Z}_{1,\bullet}^{\mathrm{inst}} = t[N]_t \tag{F.1}$$

$$\widetilde{Z}_{1,1}^{\mathrm{inst}} = (qQ + [N]_t)[N]_t \tag{F.2}$$

$$t \begin{pmatrix} \widetilde{Z}_{2,\bullet}^{\mathrm{inst}} \\ \widetilde{Z}_{1^2,\bullet}^{\mathrm{inst}} \end{pmatrix} = \begin{pmatrix} 1 & \tilde{q} + t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} qQt[N]_t \\ t^2 \begin{bmatrix} N \\ 2 \end{bmatrix}_t \end{pmatrix}, \tag{F.3}$$

$$-t^{\frac{1}{2}} \begin{pmatrix} \widetilde{Z}_{2,1}^{\mathrm{inst}} \\ \widetilde{Z}_{1^2,1}^{\mathrm{inst}} \end{pmatrix} = \begin{pmatrix} q & \tilde{q} + q(t-1) + [2]_t & \tilde{q} + t \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} q^2 Q^2 [N]_t \\ qQ[2]_t \begin{bmatrix} N \\ 2 \end{bmatrix}_t \\ t^2 [3]_t \begin{bmatrix} N \\ 3 \end{bmatrix}_t \end{pmatrix}, \tag{F.4}$$

$$\begin{pmatrix} \widetilde{Z}_{2,1^2}^{\mathrm{inst}} \\ \widetilde{Z}_{1^2,1^2}^{\mathrm{inst}} \end{pmatrix} = \begin{pmatrix} \tilde{q} + qt[2]_t + t & (\tilde{q} + q(t-1))[2]_t + [3]_t & \tilde{q} + t \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{q^2 Q^2}{t^2} \begin{bmatrix} N \\ 2 \end{bmatrix}_t \\ \frac{qQ}{t} [3]_t \begin{bmatrix} N \\ 3 \end{bmatrix}_t \\ t^2 \begin{bmatrix} 4 \\ 2 \end{bmatrix}_t \begin{bmatrix} N \\ 4 \end{bmatrix}_t \end{pmatrix},$$
$$\tag{F.5}$$

$$-t^{-\frac{1}{2}}\begin{pmatrix}\widetilde{Z}^{\text{inst}}_{2,1^3}\\ \widetilde{Z}^{\text{inst}}_{1^2,1^3}\end{pmatrix}=\begin{pmatrix}\tilde{q}+(qt+1)t & (\tilde{q}+q(t-1))[\,3\,]_t+[\,4\,]_t & \tilde{q}+t\\ 1 & [\,3\,]_t & 1\end{pmatrix}\begin{pmatrix}\frac{q^2Q^2}{t^4}[\,3\,]_t\begin{bmatrix}N\\3\end{bmatrix}_t\\ \frac{qQ}{t^2}[\,4\,]_t\begin{bmatrix}N\\4\end{bmatrix}_t\\ t^2\begin{bmatrix}5\\3\end{bmatrix}_t\begin{bmatrix}N\\5\end{bmatrix}_t\end{pmatrix}.$$
$$\tag{F.6}$$

Note that $[\,4\,]_t=(t^2+1)[\,2\,]_t$. Since $\{\tilde{q}+q(t-1)\}[\,2\,]_t$ with $\tilde{q}=q+p$ or $(1+qt)q/t+p$ is a polynomial in q, t and p with positive integer coefficients, so is $\widetilde{Z}^{\text{inst}}_{\lambda,1^s}$ for $|\lambda|=2$ and $s=0,1,2,3$.

$$-t^{\frac{3}{2}}\begin{pmatrix}\widetilde{Z}^{\text{inst}}_{3,\bullet}\\ \widetilde{Z}^{\text{inst}}_{(2,1),\bullet}\\ \widetilde{Z}^{\text{inst}}_{1^3,\bullet}\end{pmatrix}=M_{3,0}\begin{pmatrix}q^2Q^2[\,N\,]_t\\ qQt[\,2\,]_t\begin{bmatrix}N\\2\end{bmatrix}_t\\ t^3\begin{bmatrix}N\\3\end{bmatrix}_t\end{pmatrix},\qquad t\begin{pmatrix}\widetilde{Z}^{\text{inst}}_{3,1}\\ \widetilde{Z}^{\text{inst}}_{(2,1),1}\\ \widetilde{Z}^{\text{inst}}_{1^3,1}\end{pmatrix}=M_{3,1}\begin{pmatrix}q^4Q^3[\,N\,]_t\\ \frac{q^2Q^2}{t}[\,2\,]_t\begin{bmatrix}N\\2\end{bmatrix}_t\\ qQ[\,3\,]_t\begin{bmatrix}N\\3\end{bmatrix}_t\\ t^3[\,4\,]_t\begin{bmatrix}N\\4\end{bmatrix}_t\end{pmatrix},$$
$$\tag{F.7}$$

$$-t^{\frac{1}{2}}\begin{pmatrix}\widetilde{Z}^{\text{inst}}_{3,1^2}\\ \widetilde{Z}^{\text{inst}}_{(2,1),1^2}\\ \widetilde{Z}^{\text{inst}}_{1^3,1^2}\end{pmatrix}=M_{3,2}\begin{pmatrix}q^4Q^3[\,2\,]_t\begin{bmatrix}N\\2\end{bmatrix}_t\\ \frac{q^2Q^2}{t^3}[\,3\,]_t\begin{bmatrix}N\\3\end{bmatrix}_t\\ \frac{qQ}{t}[\,3\,]_t[\,4\,]_t\begin{bmatrix}N\\4\end{bmatrix}_t\\ t^3\begin{bmatrix}5\\2\end{bmatrix}_t\begin{bmatrix}N\\5\end{bmatrix}_t\end{pmatrix},\qquad t^2\begin{pmatrix}\widetilde{Z}^{\text{inst}}_{4,\bullet}\\ \widetilde{Z}^{\text{inst}}_{(3,1),\bullet}\\ \widetilde{Z}^{\text{inst}}_{2^2,\bullet}\\ \widetilde{Z}^{\text{inst}}_{(2,1^2),\bullet}\\ \widetilde{Z}^{\text{inst}}_{1^4,\bullet}\end{pmatrix}=M_{4,0}\begin{pmatrix}q^3Q^3[\,N\,]_t\\ q^2Q^2\begin{bmatrix}N\\2\end{bmatrix}_t\\ qQt[\,3\,]_t\begin{bmatrix}N\\3\end{bmatrix}_t\\ t^4\begin{bmatrix}N\\4\end{bmatrix}_t\end{pmatrix},$$
$$\tag{F.8}$$

with $M_{3,s}$ and $M_{4,0}$ in the next page. Since

$$\{\tilde{q}+q(t^2-1)\}[\,3\,]_t,\qquad \tilde{q}(\tilde{q}+1)(qt+1)+q^2(t-1)(qt+[\,2\,]_t),$$
$$\{\tilde{q}^3+(\tilde{q}(\tilde{q}+1)+q(t-1))(q(t-1)+[\,2\,]_t)[\,2\,]_t\}[\,3\,]_t,\tag{F.9}$$

with $\tilde{q}=q+p$ or $(1+qt)q/t+p$ are polynomials in q, t and p with positive integer coefficients, so is $\widetilde{Z}^{\text{inst}}_{\lambda,1^s}$ for $|\lambda|=3$ and $s=0,1,2$. The same is true for $|\lambda|=4,5$ and $|\lambda|+s\leq5$ with $\tilde{q}=q$ or $(1+qt)q/t+p$.

$$t^2\widetilde{Z}^{\text{inst}}_{2,2}(t^N;q,t)=q^6Q^3t[\,N\,]_t+(\tilde{q}+t)^2t^4(t^2+1)[\,3\,]_t\begin{bmatrix}N\\4\end{bmatrix}_t$$
$$+q^2Q^2\left\{\{2\tilde{q}t+q(qt+[\,2\,]_t)(t-1)\}q[\,2\,]_t+(\tilde{q}^2+t)^2+(qt[\,2\,]_t+1)[\,2\,]_t\right\}\begin{bmatrix}N\\2\end{bmatrix}_t$$
$$+qQt\left\{(\tilde{q}+q(t-1))(\tilde{q}+q(t-1)+[\,2\,]_t)[\,2\,]_t+\tilde{q}^2(t^2+1)+t([\,3\,]_t+1)\right\}[\,3\,]_t\begin{bmatrix}N\\3\end{bmatrix}_t.\tag{F.10}$$

Since $\{\tilde{q}+q(t-1)\}[\,2\,]_t$, $\{\tilde{q}+q(t-1)\}[\,3\,]_t$, $\tilde{q}t+(t-1)q(qt+[\,2\,]_t)$, with $\tilde{q}=(1+qt)q/t+p$ are polynomials in q, t and p with positive integer coefficients, so is $\widetilde{Z}^{\text{inst}}_{2,2}$. The same is true for $(\lambda,\mu)=(3,2)$ and $((2,1),2)$ with $(1+qt)q/t+p$.

$$
{}^{t}M_{3,0} = \begin{pmatrix} qt & 0 & 0 \\ \tilde{q}(\tilde{q}+1) + q(t-1) + t & 1 & 0 \\ \tilde{q}^3 + \tilde{q}(\tilde{q}+1)t[\,2\,]_t + t^3 & \tilde{q} + t[\,2\,]_t & 1 \end{pmatrix}.
$$

$$
{}^{t}M_{3,1} = \begin{pmatrix} q^2 & 0 & 0 \\ \tilde{q}(\tilde{q}+1)(qt+1) + q^2(t-1)(qt+[\,2\,]_t) + (q[\,2\,]_t+1)t & qt+1 & 0 \\ \tilde{q}^3 + (\tilde{q}(\tilde{q}+1) + q(t-1))(q(t-1)+[\,2\,]_t)[\,2\,]_t + t[\,3\,]_t & \tilde{q} + (q(t-1)+[\,2\,]_t)[\,2\,]_t & 1 \\ \tilde{q}^3 + \tilde{q}(\tilde{q}+1)t[\,2\,]_t + t^3 & \tilde{q} + t[\,2\,]_t & 1 \end{pmatrix}.
$$

$$
{}^{t}M_{3,2} =
$$
$$
\begin{pmatrix} \tilde{q}(\tilde{q}+1) + q^2t^2 + q(t-1) + t & t^{-3} & 0 \\ \tilde{q}^3 + \tilde{q}(\tilde{q}+1)(q(t^2+t-1)+[\,2\,]_t)[\,2\,]_t + qt(t-1)(q[\,2\,]_t(qt+[\,2\,]_t)+1) + qt^3(t+2) + t[\,3\,]_t & \tilde{q} + q(t^2+t-1)[\,2\,]_t + [\,2\,]_t^2 & 1 \\ \tilde{q}^3 + (\tilde{q}(\tilde{q}+1) + q(t-1))(q(t^2-1)+[\,3\,]_t) + t(t^2+1) & \tilde{q} + q(t^2-1) + [\,3\,]_t & 1 \\ \tilde{q}^3 + \tilde{q}(\tilde{q}+1)t[\,2\,]_t + t^3 & \tilde{q} + t[\,2\,]_t & 1 \end{pmatrix}.
$$

$$
{}^{t}M_{4,0} =
$$
$$
\begin{pmatrix} q^3t & 0 & 0 & 0 & 0 \\ \tilde{q}^4 + \tilde{q}^2 + \tilde{q}[\,3\,]_{\tilde{q}}(q[\,2\,]_t+1)t + q^2(t-1)[\,2\,]_t(qt+[\,2\,]_t) + (q[\,2\,]_t+1)t^2 & \tilde{q} + [\,2\,]_t tq + t & 1 & 0 & 0 \\ q^2t(\tilde{q}[\,3\,]_{\tilde{q}} + (\tilde{q}^2+1)t) + ((t-1)q+t)(\tilde{q}[\,3\,]_{\tilde{q}} + (t-1)q)[\,2\,]_t + t^3 & \tilde{q}^2 + [\,2\,]_t(\tilde{q} + (t-1)q + t) & \tilde{q} + t & 1 & 0 \\ \tilde{q}^6 + \tilde{q}[\,3\,]_{\tilde{q}}(\tilde{q}^2+t^2)t[\,3\,]_t + (\tilde{q}^4+\tilde{q}^2)(t^4+t^2) + t^6 & \tilde{q}^3 + \tilde{q}(\tilde{q}+1)t[\,3\,]_t + \tilde{q}t^2(t^2+1) + t^3[\,3\,]_t & \tilde{q}^2 + \tilde{q}t[\,3\,]_t + t^4 + t^2 & \tilde{q} + t[\,3\,]_t & 1 \end{pmatrix}.
$$

$$t^{\frac{5}{2}}\widetilde{Z}^{\text{inst}}_{(2,1),2}(t^N; q, t) = (\tilde{q} + t[\,2\,]_t)(\tilde{q} + t)(t^2+1)t^5[\,5\,]_t \begin{bmatrix} N \\ 5 \end{bmatrix}_t + \frac{q^4 Q^3}{t}\left\{\tilde{q} + q(t-1) + q^2 t^2 + [\,2\,]_t\right\}[\,2\,]_t \begin{bmatrix} N \\ 2 \end{bmatrix}_t$$

$$+ \frac{q^2 Q^2}{t}\left\{\tilde{q}^2 + q\left\{\tilde{q} + q^2 t^2 + (qt+1)[\,2\,]_t\right\}(t-1) + \tilde{q}\left\{qt^2(t+2) + [\,3\,]_t + 2t\right\} + ([\,3\,]_t + qt^2[\,2\,]_t)[\,2\,]_t\right\}[\,3\,]_t \begin{bmatrix} N \\ 3 \end{bmatrix}_t$$

$$+ qQ\left\{\left\{\tilde{q}^2 + q(q[\,2\,]_t(t-1) + \tilde{q}(t+2) + [\,2\,]_t{}^2)(t-1)\right\}[\,3\,]_t + \tilde{q}([\,3\,]_t{}^2 + [\,4\,]_t) + t([\,3\,]_t{}^2 + [\,2\,]_t)\right\}t(t^2+1)[\,2\,]_t \begin{bmatrix} N \\ 4 \end{bmatrix}_t.$$

$$t^{\frac{5}{2}}\widetilde{Z}^{\text{inst}}_{3,2}(t^N; q, t) = q^{10}Q^4 \begin{bmatrix} N \\ 1 \end{bmatrix}_t t + \left\{\tilde{q}^4 + \tilde{q}^3 t(t+2) + \tilde{q}^2 t[\,2\,]_t{}^2 + \tilde{q}t^2(2t+1) + t^4\right\}(t^2+1)t^5[\,5\,]_t \begin{bmatrix} N \\ 5 \end{bmatrix}_t$$

$$+ \frac{q^4 Q^3}{t}\left\{\tilde{q}^3 + q\left\{\tilde{q}(\tilde{q}+2) + ((q^3 t + 1)(qt + [\,2\,]_t) + q^2 t[\,2\,]_t)\right\}(t-1)\right.$$

$$\left. + \tilde{q}^2(q^2 t^2 + t + 2) + \tilde{q}(2q^2 t^2 + 2t + 1) + (q^2 t^2[\,2\,]_t + [\,3\,]_t)\right\}[\,2\,]_t \begin{bmatrix} N \\ 2 \end{bmatrix}_t$$

$$+ \frac{q^2 Q^2}{t}\left\{\tilde{q}^4 + q\left\{qt(qt + [\,2\,]_t) + [\,2\,]_t\right\}(\tilde{q}^2 + q^2(t-1))(t-1)[\,2\,]_t\right.$$

$$+ q\left\{\tilde{q}^2(\tilde{q}+1)[\,2\,]_t + \tilde{q}((2qt(qt + [\,2\,]_t) + 1)[\,2\,]_t + 2t) + ((2q^2 t^2 + q[\,2\,]_t{}^2 + 2)t[\,2\,]_t + 1)\right\}(t-1)$$

$$+ \left\{\tilde{q}^3(qt^2(t+2) + [\,3\,]_t + 2t) + \tilde{q}^2([\,3\,]_t + [\,2\,]_t + qt^2(t+2))[\,2\,]_t + \tilde{q}((2t+1)[\,3\,]_t + qt^3(2t+5)) + (qt^2[\,2\,]_t{}^2 + [\,4\,]_t + t)t\right\}\right\}[\,3\,]_t \begin{bmatrix} N \\ 3 \end{bmatrix}_t$$

$$+ qQ\left\{q^3[\,2\,]_t[\,3\,]_t(t-1)^3 + q^2(\tilde{q}(\tilde{q}+2) + [\,2\,]_t)[\,2\,]_t[\,3\,]_t(t-1)^2 + q\left\{\tilde{q}^3(t+2) + \tilde{q}(2[\,3\,]_t + t) + \tilde{q}^2[\,2\,]_t(t+2) + t([\,3\,]_t + 1)\right\}[\,3\,]_t(t-1)\right.$$

$$\left. + \tilde{q}^4[\,3\,]_t + \tilde{q}^3([\,3\,]_t{}^2 + [\,4\,]_t) + \tilde{q}([\,4\,]_t[\,2\,]_t + [\,3\,]_t(t^2 + 2))t + ([\,4\,]_t + t)[\,2\,]_t{}^2\tilde{q}^2 + ([\,5\,]_t + t[\,2\,]_t)t^2\right\}t[\,4\,]_t \begin{bmatrix} N \\ 4 \end{bmatrix}_t.$$

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